BANACH SPACE REPRESENTATIONS OF BANACH ALGEBRAS

by

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CERTIFICATE

This is to certify that the work embodied in the thesis entitled, BANACH SPACE REPRESENTATIONS OF BANACH ALGEBRAS, by V. Muruganandam, has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

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CONTENTS

CHAPTER 0

INTRODUCTION

This thesis grew out of our attempts to study the Banach space representations of a Banach algebra, as the title suggests. We carry out our investigations by stages first focussing our attention on commutative Banach algebras and then on non-commutative Banach algebras but often digressing to cover the areas related to them. In the case of commutative B-algebra, the rich structure theory is applied to understand the nature of the representations. In the non-commutative case we define and discuss the notion of weak containment among the Banach space representations of a Banach algebra, in terms of coordinate functionals associated with them. Applying the results we obtained here to $L^1(G)$; G a locally compact group, we construct a host of function algebras consisting of coordinate functionals belonging to certain representations of G.

We carry out an extensive study of certain invariant subspaces of the contragradient representation of regular representation of a Banach algebra A, and their duals. These duals are actually quotient algebras of A^{**}, the bidual of A containing A isomorphically. These studies are applied for ascertaining weakly compact multipliers of A. The following may deviate a little from the main theme. Certain

aspects of representation theory lead us to discuss non-standard hull of a representation and this prompted us to study non-standard hull of a Banach algebra. We prove some theorems indicating certain properties, it shares with original Banach algebra with which we started.

A detailed description of the contents and organisation of the materials covered are as follows. Chapter 1 contains a glossary of symbols, basic definitions and relevant examples alongwith a brief summary of elementary facts about the non-standard analysis, which we need in the sequel. In Chapter 2, we construct non-standard hull $\hat{A}$ of a Banach algebra A and discuss some of its properties. If A is commutative, we find out the relations between the maximal ideal spaces of A and $\hat{A}$, and draw some interesting corollaries for specific classes of Banach algebras. Then we define non-standard hull of a B-space representation of a B-algebra and that of a locally compact group.

Domar and Lindahl [13] studied the Banach space representation (π, X) of a commutative Banach algebra by considering what is known as spectrum of π. In Chapter 3, we pursue their line of approach to explore the representations of a commutative Banach algebra A. As Arveson and others had done on abelian group representations, we define spectral subspaces associated with π and systematically study them. Making use of these subspaces we capture the spectrum of π. We give a necessary and sufficient condition for a $\xi \neq 0$ in X to be a common eigenvector for all operators $\pi(x)$; x ranging over A, in terms of these spectral subspaces.

We obtain many other results and include several concrete examples illustrating the theory. Assuming the spectrum of π to be discrete we decompose (π, X) into its minimal subrepresentations.

Chapter 4 is devoted for the applications of the theory we developed in Chapter 3. It contains two major theorems, each one is entirely different from the other in its flavour. One gives a necessary and sufficient condition for A to have a discrete maximal ideal space in terms of multipliers. The other proves that a norm closed, A- invariant subspace of the dual A^* is necessarily finite dimensional if it is reflexive.

It is well known that the bidual A^{**}, with any of the Arens products is a Banach algebra extending the multiplication of A. But it is too large to retain some of the basic properties of A. This has naturally lead us to consider certain quotient algebras of A^{**} which contain A isomorphically. These algebras are duals of certain A- invariant subspaces of A^* which are known as introverted subspaces. Specific examples of introverted subspaces were studied by various authors and A.T. Lau [32] studied them for a particular B-algebra namely $A_2(G)$. The aim of Chapter 5 is to present a unified approach to the study of introverted subspaces on a arbitrary B-algebra. Starting with a commutative B-algebra A one likes to know how far the commutativity of A can be extended to these quotient algebras. We give a definitive solution to this problem generalizing the result obtained by A.T. Lau [32] for $A_2(G)$. For a Banach algebra A with bounded right approximate identity we give a functional characterization for the algebra of operators on an introverted subspace X commuting with the action of A.

As an application of our results we prove that there is no nontrivial weakly compact multiplier on a commutative M-regular N algebra (Ref. 1.2.3 for the definition) whose maximal ideal space contains no isolated points. We conclude this chapter by proving a theorem which gives various characterizations for a commutative M-regular N algebra with discrete maximal ideal space.

In Chapter 6, we study the notion of weak containment among the Banach space representations of a Banach algebra A. With (π, X) there arise two subspaces in A^*.

$$W_\pi = \{f \in A^* : \text{there exists a } K > 0 \text{ such that } |f(a)| \leq K \, \|\pi(a)\| \text{ for every } a \in A\}.$$

If $\pi_{\xi,\eta}$ denotes the bounded linear functional on A defined by $a \to (\pi(a)\,\xi, \eta)$ for any $\xi \in X$, $\eta \in X^*$, then define

$$T_\pi = \{ \sum_{n=1}^{\infty} \pi_{\xi_n, \eta_n} : (\xi_n) \subseteq X,\ (\eta_n) \subseteq X^* \text{ and } \sum_{n=1}^{\infty} \|\xi_n\| \, \|\eta_n\| < \infty \}.$$

It can be easily seen that $T_\pi \subseteq W_\pi$. But then, trivial examples show that T_π need not be equal to W_π. Then one can ask whether there exists another representation (π', X') which is nicely related to (π, X) such that $W_\pi = T_{\pi'}$. The main theorem of Cowling and Fendler [8] gives an affirmative answer to this question. We shall give a simpler and modified proof to this theorem using non-standard methods, but essentially following their ideas.

We define and discuss B-lattice representations of a Banach-algebra lattice and study such representations of a locally compact group with special emphasis to p-representations i.e. when the

representation space is abstract L^p-space. Motivated by the construction of Fourier-stiltjes algebra $B(G)$ for a locally compact group G due to Eymard [15] we construct a host of function algebras $F_p(G)$ on G consisting of coordinate functions belonging to p-representations of G.

CHAPTER 1

PRELIMINARIES AND NOTATIONS

In this chapter we give a list of notations and conventions. We collect the basic definitions and facts which we may need. We discuss some relevant examples. Finally we summarize certain definitions, terminologies and basic ideas used in the framework of non-standard analysis.

Glossary of Special Symbols:

$\emptyset$	-	Empty set.
$\mathbb{R}$	-	The field of real numbers.
$\mathbb{C}$	-	The field of complex numbers.
$\mathbb{N}$	-	Positive integers.
A^c	-	Complement of a set A.
f/A	-	The function f restricted to A.
Int A	-	Interior of A.
$\bar{A}$	-	Closure of A.

If $f: S \to \mathbb{C}$, where S is a topological space, then the set $\{x \in S: f(x) \neq 0\}^-$ is called the support of f and is denoted by supp f. χ_A denotes the characteristic function of A.

We write a locally compact T_2 group as lc group and a locally compact T_2 abelian group by lca group.

Section 1.1 Banach Spaces:

1.1.1:

By a B-space we mean a Banach space over $\mathbb{C}$ and we usually denote it by X. If X,Y are any two B-spaces then, L(X,Y) denotes the space of all bounded linear operators from X into Y. If Y = X, we denote this by L(X) and if Y = $\mathbb{C}$, we denote this by X^*.

If X, X^* is a dual pair, then (,) - usually denotes the bilinear form on X x X^* given by $(\xi,\eta) = \eta(\xi)$ for $\xi \in X$ and $\eta \in X^*$. A subset V in X^* is called total if $(\xi,\eta) = 0$ for every $\eta \in V$ then $\xi = 0$. Let V be a total set in X^*. Then the topology defined by the seminorms $\{\xi \to |(\xi,\eta)| : \eta \in V\}$ on X is denoted by $\sigma(X,V)$. In particular, if V = X^*, this is called weak topology and $\sigma(X^*,X)$ - topology is known as weak-* topology.

Let V be a subset in X.

[V] - subspace spanned by V.

co(V) - convex hull of V.

$V^{\perp}$ - annihilator of V i.e. $\{\eta \in X^* : (\xi,\eta) = 0 \ \forall \ \xi \in V\}$.

For any $f \in X^*$, let us denote the kernel of f by Ker f.

Let $1 \leq p \leq \infty$. By an L^p-space we mean $L^p(S,\Sigma,m)$ for some measure space (S,Σ,m). In particular if S is a lc group G, and m is the left invariant Haar measure, then it will be denoted by $L^p(G)$. By M(G) we denote the space of all finite regular Borel measures on a lc group G. If S is a locally compact space, then $C_0(S)$ denotes the space of all continuous functions which vanish at infinity.

1.1.2 Direct Sum of B-spaces:

Let $1 \leq p \leq \infty$. Let $\{X_i, \| \ \|_i\}_{i\varepsilon I}$ be a collection of B-spaces. By $\ell^p(I,X_i)$ we mean the space $\{\xi = (\xi_i)_{i\varepsilon I}: \sum_{i\varepsilon I} \|\xi_i\|^p < \infty\}$ equipped with the natural norm $\|\xi\| = \{\sum_{i\varepsilon I} \|\xi_i\|^p\}^{1/p}$.
If $X_i = X \ \forall \ i$ then we denote this by $\ell^p(I,X)$.

Note that the B-space $\ell^q(I,X^*)$ is contained in the dual of $(\ell^p(I,X)$, and if $1 < p < \infty$ then $\ell^q(I,X^*)$ equals $(\ell^p(I,X))^*$ where $p^{-1} + q^{-1} = 1$.

1.1.3 B-lattices:

For any B-lattice X the positive cone is usually denoted by X_+. By a complex B-lattice X we mean the complexification of a real B-lattice i.e. there exists a real B-lattice Y such that every $z \varepsilon X$ is written as $x+iy$, $x,y \varepsilon Y$, and $|z| = \{|x|^2+|y|^2\}^{1/2}$. We say that an operator T from X into Y is positive if $T(X_+) \subseteq Y_+$.

A B-lattice X is called Abstract p-space if for every x,y in X, such that $x \wedge y = 0$, $\|x+y\|^p = \|x\|^p + \|y\|^p$ holds. It is well known that an abstract p-space can be concretely realized as $L^p(S,\Sigma,m)$. We call these spaces simply by p-spaces.

Notice that if $\{X_i, \| \ \|_i\}_{i\varepsilon I}$ is a collection of p-spaces for some $p; 1 \leq p \leq \infty$, then $\ell^p(I,X_i)$ is again a p-space with a natural order defined by: $\{\xi_i\} \geq 0$ if for every $i \varepsilon I$, $\xi_i \geq 0$.

We briefly describe the tensor product of B-lattices. For a detailed account refer Schaefer [39].

Let X,Y be B-lattices and T be in L(X,Y). We say that T factors through a 1-space Z with bound k if there exists operators, $T_1: X \to Z$ and $T_2: Z \to Y$ such that

(i) T_1 is positive and $\| T_1 \| \leq k$,

(ii) $\| T_2 \| \leq 1$,

(iii) $T = T_2 o T_1$ hold.

Such operators form a vector subspace $L^{\ell}(X,Y)$ of L(X,Y). If $\| T \|_{\ell}$ is infirmum of all k's as above for any $T \in L^{\ell}(X,Y)$, then it defines a norm with which $L^{\ell}(X,Y)$ is a B-space.

Now the natural map $\beta: X \otimes Y \to L(X^*,Y)$ defined by $\beta(x \otimes y)(x^*) = (x,x^*)y$ actually takes $X \otimes Y$ into $L^{\ell}(X^*,Y)$ with $\| \beta(x \otimes y) \|_{\ell} = \| x \| \; \|y\| \quad \forall\, x \in X;\ y \in Y$. Completion of $X \otimes Y$ with respect to the above norm $\| \; \|_{\ell}$ is again a B-lattice whose positive cone is the norm closure of the convex cone spanned by elements of $X_+ \otimes Y_+$ and this B-lattice is usually denoted by $X \otimes_{\ell} Y$. Furthermore, if $1 < p < \infty$, and X,Y are p-spaces then so is $X \otimes_{\ell} Y$ and if X and Y are 1-spaces then so is $X \otimes_{\ell} Y$ and $\| \; \|_{\ell}$ is none other than projective norm.

Section 1.2 Banach algebras:

1.2.1

The following notations are fixed throughout this thesis.

A - Any Banach algebra over $\mathbb{C}$. We usually write a Banach algebra as B-algebra. A is always assumed to be right faithful, i.e. if for some b in A ab = 0 for every $a \in A$ then b = 0.

A_e - The Banach algebra got by adjoining identity to A i.e., $A_e = \{(x,r): x \varepsilon A,\ r \varepsilon C\}$.

If A is commutative, then

$\Delta(A)$ - The maximal ideal space or spectrum of A, i.e. the space of all non-zero complex homomorphisms on A with Gelfand topology.

$\hat{x}$ - The Gelfand transform of x on $\Delta(A)$ defined by $\hat{x}(\tau) = \tau(x)$

$\hat{A}$ - Image of A under Gelfand transform

C $= \{x \varepsilon A: \text{supp } \hat{x} \text{ is compact}\}$.

Let us observe that, $\Delta(A_e) = \Delta(A) \cup \{\tau_\infty\}$ where τ_∞ maps any element (x,r) to r.

We say that A has got 1-right approximate identity if there exists a net $\{e_i\}$ in A such that, $\|e_i\| \leq 1 \ \forall\ i$, $\lim_i \|ae_i - e_i\| = 0 \ \forall\ a \varepsilon A$. If it is two sided, then we simply call as approximate identity. $\{e_i\}$ stands for approximate identity, unless specifically mentioned otherwise.

<u>1.2.2</u>

Let us discuss some terminologies in the context of commutative B-algebras.

We say that A is semisimple if the Gelfand transform is injective i.e., for any $x \varepsilon A$, if $\tau(x) = 0 \ \forall\ \tau \varepsilon \Delta(A)$ then $x = 0$.

A is called regular if for any closed set E and τ in $\Delta(A)$ such that $\tau \notin E$, there exists $x \varepsilon A$ such that $\hat{x}(\tau) = 1$ and $\hat{x}/E = 0$. Moreover, if $\|x\| \leq M$ for some $M > 0$, independent of τ and E, then A is called as M-regular.

We shall give a number of examples of M-regular algebras. We do not know whether every regular commutative Banach algebra is automatically M-regular for some $M > 0$.

A is called Tauberian if C is norm dense in A.

For every closed ideal I, $\{\tau \in \Delta(A): \tau(x) = 0$ for every $x \in I\}$ is a closed set in $\Delta(A)$ and is called as hull of I. We denote it by $h(I)$.

For any closed set E in $\Delta(A)$ we denote the ideal $\{x \in C: \hat{x}$ vanishes on a neighbourhood of $E\}$ by $J_o(E)$ and it is well known that the closure of $J_o(E)$ is the smallest among all the closed ideals I such that $h(I) = E$, for a semisimple regular B-algebra A. We say that E is a set of spectral synthesis or simply S-set if $J_o(E)^-$ is the only closed ideal whose hull is E.

<u>1.2.3 Definition</u>:

We say that a commutative B-algebra A is N-algebra if A is semisimple, Tauberian and every singleton set $\{\omega\}$ in $\Delta(A)$ is a S-set. If in addition, it is M-regular, then we call it as M-regular N algebra.

We give some examples of M-regular N algebras.

1. $C_o(S)$ is an 1-regular, N algebra.
2. In particular $L^\infty(S,\Sigma,m)$ for any measure space (S,Σ,m) and AP(G), the algebra of all almost periodic functions on a lca group G are 1-regular N algebras.
3. Let $A = L^1(G)$: G a lca group. Then A is a 2-regular N algebra. Owing to the theorem 2.6.3 of Rudin [38] we conclude that $L^1(G)$ is 2-regular. The fact that every singleton is a S-set is just Beurling-Kaplansky's Theorem.

4. Beurling Algebras: Let G be a lca group. Let ω be a real valued function on G such that

$$\omega(x) \geq 1 \quad \forall \ x \in G$$

$$\omega(xy) \leq \omega(x)\,\omega(y) \quad \forall \ x, y \in G \quad \text{and}$$

ω is measurable and locally bounded. Then the algebra $\{f : f.\ \omega \in L^1(G)\}$ is called as the Beurling algebra and is denoted by $L^1_\omega(G)$, and it is a commutative 2-regular N algebra if the weight function ω satisfies the following Shilov's condition:

(i) $\omega(x^n) = O(|n|^\alpha);\ |n| \to \infty$ for some $\alpha > 0$, α depends on x.

(ii) $$\liminf_{|n| \to \infty} \frac{\omega(x^n)}{|n|} = 0, \quad \forall \ x \in G.$$

Refer Reiter [35; Chapter 3,6] for more details.

5. In the following paragraph we briefly describe the socalled Figa-Talamanca-Herz algebras $A_p(G)$; G a lc group and $1 < p < \infty$ and assert that they are 1-regular N algebras.

$A_p(G)$ is the B-algebra of all functions u of the form

$u = \sum_{n=1}^{\infty} f_n * \check{g}_n$, where, $\{f_n\} \subseteq L^p(G)$; $\{g_n\} \subseteq L^q(G)$ such that

$$\sum_{n=1}^{\infty} \| f_n \| \, \| g_n \| < \infty.$$

Here $\check{g}$ denotes the function $t \to g(t^{-1})$: $t \in G$.

$A_p(G)$ is actually a function algebra with point-wise multiplication and the norm of u, for any $u \in A_p(G)$, is given by $\text{Inf}\ \{\Sigma \| f_n \| \, \| g_n \|\}$ where the infirmum is taken over all the representations of u. It is regular, Tauberian and

its maximal ideal space is identified with G, with Gelfand transform being identity. Here every singleton set in G is a S-set. For a detailed discussion refer Herz [27,part 3] and Eymard [16, p.13]. In what follows we shall give an explicit proof that these are 1-regular, since we donot find a proof already existing in the literature.

Claim: There exists a $u \in A_p(G)$ such that $u/U^c = 0$; $u(e) = 1$ and $\|u\| \leq 1$ for every compact neighbourhood U of e.

Let V be a symmetric neighbourhood of e such that $V^2 \subseteq U$.

Consider $f = m(V)^{-1/p} \chi_V$ where m is the left invariant Haar measure on G and χ_V is the characteristic function on V.

Let $g = f^{p-1}$. Then $f \in L^p(G)$; $g \in L^q(G)$ and $\|f\|_p = \|g\|_q = 1$.

Consider $u = f * \check{g}$. Then $u(e) = 1$ and $\|u\| \leq \|f\|_p \|g\|_q = 1$, again observe that $u/U^c = 0$.

Since $A_p(G)$ is invariant under translations and, $\|{}_su\| = \|u\|$ where ${}_su$ is left translate of u by s, the result is true for any $s \in G$. Thus $A_p(G)$ is 1-regular.

Remark: Refer Loomis [33] or Rickart [36] for the unexplained terminologies if any.

1.2.4

In this paragraph we discuss about B-space representations of lc groups and B-algebras.

Let G be a locally compact group. We say that (π, X) is a B-space representation if

(i) X is a B-space

(ii) $\pi: G \to L(X)$ is a map satisfying $\pi(st) = \pi(s)\circ\pi(t)$ and the mapping $(t, \xi) \to \pi(t)\xi$ is continuous from $G \times X$ into X.

We say that (π,X) is uniformly bounded if there exists $M > 0$ such that, $\|\pi(t)\| \leq M$ for every $t \in G$.

We say that any two representations (π_1, X_1) and (π_2, X_2) are equivalent if there exists a $T \in L(X,Y)$ and T is invertible such that $T\circ\pi_1(t) = \pi_2(t)\circ T$ for every $t \in G$.

We renorm $(X, \|\ \|)$ if necessary as

$$\|\xi\|' = \sup_{t\in G} \{\|\pi(t)\,\xi\|\}$$

so that, the representation π of G on $(X, \|\ \|)$ and the representation π of G on $(X, \|\ \|')$ are equivalent. With the new norm, $\|\ \|'$, we have $\|\pi(t)\| \leq 1$ $\forall\ t \in G$ and consequently $\|\pi(t)\| = 1$ $\forall\ t \in G$.

We henceforth make it a convention that <u>for any B-space representation (π,X), $\|\pi(t)\| = 1$ for every $t \in G$</u>.

The usual representation of G on $L^p(G)$ defined by $f \to {}_{(s^{-1})}f$ will be denoted as λ_p and we call it as left-p-regular representation of G.

Consider a B-algebra A. (π,X) is called a B-space representation of A if

(i) X is a B-space

(ii) $\pi: A \to L(X)$ is an algebra homomorphism.

We say that (π, X) is a bounded B-space representation if there exists $M > 0$ such that $\|\pi(x)\| \leq M\|x\|$ for every $x \in A$.

Let (π, X) be a bounded B-space representation of A. As in the case of groups we renorm X if necessary, as

$$\|\xi\|' = \text{Max} \left\{ \sup_{\|x\| \leq 1} (\|\pi(x)\xi\|) \; ; \; \|\xi\| \right\}$$

so that, π is equivalent to the representation π' of A acting on $(X, \| \, \|')$ which satisfies, $\|\pi'(x)\| \leq \|x\| \;\; \forall \, x \in A$.

Henceforth we make it a convention that <u>for any B-space representation (π, X) of A $\|\pi(x)\| \leq \|x\| \;\; \forall \, x \in A$</u>.

The usual left regular representation of A acting on A by $b \to ab$; $a, b \in A$ is denoted by λ.

We say that (π, X) is non degenerate if the closure of $[\pi(A)X]$ equals X.

<u>1.2.5</u>: Fix $1 \leq p < \infty$. If $\{\pi_i, X_i\}_{i \in I}$ is a collection of B-space representations then by

$$\{ \sum_{i \in I}^{\oplus} \pi_i ; \; \ell^p(I, X_i) \}$$

we mean the direct sum of the B-space representation.

If in particular $\pi_i = \pi$ and $X_i = X$ for every $i \in I$ then the above representation will simply be denoted by $(\pi^{(p)}, \ell^p(I, X))$.

<u>1.2.6</u>:

Let $\tilde{A}$ denote the B-space A with a new multiplication, as $x \circ y = yx \;\; \forall \, x \in A$. Then $\tilde{A}$ is a Banach algebra which is called as reverse algebra. If A is commutative then $A = \tilde{A}$.

If (π,X) is a representation of A, then, $a \to \pi(a)^*$ is a representation of $\tilde{A}$ with representation space X^*. This representation is known as contragradient representation of (π,X) and is denoted by $(\bar{\pi},X^*)$.

1.2.7:

Let $L_\pi(X)$ denote the algebra of all bounded operators T on X such that $T \circ \pi(x) = \pi(x) \circ T \quad \forall\, x \in A$.

Notice that $L_\lambda(A)$ is none other than the algebra of all left multipliers, $M_\ell(A)$.

Finally, we remark that with our conventions as above, the B-space representations of lc group G are in one-one correspondence with non-degenerate B-space representations of its group algebra $L^1(G)$.

Section 1.3:

Let $\mathcal{M}$ denote a super structure or standard universe, containing those objects which we want to study. The standard individuals or basic elements in $\mathcal{M}$ is denoted by $\mathcal{S}$. Then there exists another universe $^*\mathcal{M}$ a non-standard enlargement and an imbedding $*$ from $\mathcal{M}$ into $^*\mathcal{M}$. $*$ takes $\mathcal{S}$ into $^*\mathcal{S}$, the non-standard individuals. We discuss some of the terminologies involved in this non-standard superstructure and identify some of the properties it enjoys.

$*$ transform of a set E in $\mathcal{M}$ is usually denoted by $^*E$. Since $*$ transform of a function f with domain E extends f from E to $^*E$, it is natural to denote $^*f$ by f itself. We do so.

If α is a sentence (i.e. a formalized mathematical assertion) in the language of $\mathcal{M}$ i.e., standard language, then the symbol $\models \alpha$ indicates that α is true in $\mathcal{M}$, and similarly $* \models \alpha$ for any α in $^*\mathcal{M}$ indicates that α is true in the non-standard language.

We state the crucial Transfer principle.

<u>1.3.1</u>: Transfer Principle

If α is a sentence in $\mathcal{M}$, let us denote the $*$-transform of α by $^*\alpha$. Then, α is true in $\mathcal{M}$ if and only if $^*\alpha$ is true in $^*\mathcal{M}$. i.e. $\models \alpha$ if and only if $* \models \alpha$.

We remark that the constants belonging to $\mathcal{S}$ in any sentence α will not get changed if we apply $*$ to α. In particular, if α involves only constants belonging to $\mathcal{S}$ then, $^*\alpha = \alpha$.

We always assume that $\mathbb{R}$, $\mathbb{C}$, are contained in $\mathcal{S}$. The non-standard enlargement $^*\mathbb{R}$ is called as the space of hyper reals. Any $p \in {}^*\mathbb{R}$, is called infinitesimal if $|p| \leq r$ for every standard real nonzero r. The set of all infinitesimals is called the monad of 0, and is denoted by $\mu(0)$.

If $p, q \in {}^*\mathbb{R}$ we say that p is infinitesimally near to q if $p-q \in \mu(0)$. Notation: $p \approx q$.

We say that p is finite if there exists r in $\mathbb{R}$ such that $p - r \in \mu(0)$ is $p \approx r$, and r is called as standard part of p and is usually denoted by st(p).

An object S in $^*\mathcal{M}$ is said to be internal if there exists a standard set E in $\mathcal{M}$ so that $S \in {}^*E$.

1.3.2: Definition

The non-standard universe $^*\mathfrak{M}$ is said to be $\mathcal{K}$-saturated, where $\mathcal{K}$ is an uncountable cordinal, if for a given family $\{S_i\}_{i\varepsilon I}$ of internal sets, contained in a fixed internal set T, the hypothesis that cardinality of $I < \mathcal{K}$ and $\{S_i\}_{i\varepsilon I}$ has finite intersection property will imply $\bigcap_{i\varepsilon I} S_i$ is not empty.

1.3.3: Concurrence Theorem:

A relation P is called concurrent in $\mathfrak{M}$ if whenever, $a_1, \ldots, a_n$ belongs to domain of P then there exists a $b \varepsilon \mathfrak{M}$ such that, $\langle a_i, b\rangle \varepsilon P \quad i = 1, 2, \ldots, n$.

Let P be a concurrent relation in $\mathfrak{M}$. Then there exists an element b in $^*\mathfrak{M}$ such that, $\langle {}^*a, b\rangle \varepsilon {}^*P$ for every a in the domain of P.

1.3.4: Q-Topology:

Let (S, Ω) be a topological space. Consider the non-standard enlargement $^*S$. We will give a topology on $^*S$ which is known as Q-topology.

Consider $^*\Omega$. The elements of $^*\Omega$ are internal sets. Observe that $^*\Omega$ contains empty set and $^*S$. The intersection of any two elements of $^*\Omega$ will again be an element in $^*\Omega$, but the union of arbitrary elements of $^*\Omega$ need not be internal. These elements of $^*\Omega$ is a basis for a topology which is called as Q-topology (See Robinson [37; 4.2.8]).

Remarks:

1. Davis [9] is a pleasant book for learning the elementary facts about non-standard analysis.

2. Standard references for the above discussions are the books of Davis [9], Robinson [37] and Stroyan and Luxemberg [40].

CHAPTER 2

NON-STANDARD HULL OF A BANACH ALGEBRA

Apart from the fact that non-standard hulls of a B-space are proving to be quite useful in understanding the local properties of Banach spaces, they themselves are becoming objects of study in their own right. The survey article by Henson and Moore [26] and the references cited therein will substantiate our claim. In this chapter we initiate the study of non-standard hull of a Banach algebra.

In Section 2.1 we recall the construction of non-standard hull $\hat{X}$ of a B-space X and some of its basic properties, In Sec.2.2 we define non-standard hull $\hat{A}$ of a B-algebra A and study some of its properties. In particular, if A is commutative then we discuss the relations between maximal ideal spaces of A and $\hat{A}$. In corollary 2.2.5, we prove that if $A = C_o(S)$, then there exists a locally compact space $\hat{S}$ such that $\hat{A} \cong C_o(\hat{S})$ and $^*S$ is topologically imbedded as a dense subset in $\hat{S}$. In Sec. 2.3, we define non-standard hull of a representation of a Banach algebra with particular reference to group algebras which will be needed for later use.

Section 2.1:

Let $\mathcal{M}$ be a superstructure with $\mathcal{S}$ standard individuals and $^*\mathcal{M}$ be a non-standard enlargement with $^*\mathcal{S}$, non-standard individuals. Assume $^*\mathcal{M}$ is $\mathcal{K}$-saturated for some uncountable

cardinal $\mathcal{K}$. (Refer Sec. 1.3 for the basic definitions).

Let X be a B-space contained in $\mathcal{S}$. We now construct non-standard hull of X. By transfer principle $^*X$ is a linear space over $^*\mathbb{C}$. We say $x \in {}^*X$ is finite if $\|x\|$ is finite. (As usual we fail to distinguish between norm functions on X and $^*X$). Denote this subspace of all finite elements by fin $(^*X)$. An element $x \in {}^*X$ is called as infinitesimal if $\|x\| \approx 0$. If $\mu(0)$ denotes the set of all infinitesimals then it is a subspace of fin $^*X$.

Consider the quotient space $\hat{X} = \frac{\text{fin } (^*X)}{\mu(0)}$. $\hat{X}$ is a linear space over $\mathbb{C}$ and is called a non-standard hull of X. Let γ designate the quotient map from fin $(^*X)$ into $\hat{X}$. Since finite points are near standard in $^*\mathbb{R}$, st$(\|x\|)$ exists for every $x \in$ fin $(^*X)$. We norm $\hat{X}$ by taking $\|\gamma(p)\| = \text{st } \|p\| \ \forall \ p \in \text{fin } (^*X)$. Then $\hat{X}$ is a normed linear space. Since $^*\mathcal{M}$ is $\mathcal{K}$-saturated, $\hat{X}$ is complete and so $\hat{X}$ is actually a Banach space. Hence non-standard hull $\hat{X}$ of a B-space X, is again a B-space and X is isometrically imbedded in $\hat{X}$.

We list out some of the basic properties. They can be either easily verified or found in Henson and Moore [26].

<u>2.1.1</u>: Proposition:

(i) If X is a B-lattice then so is $\hat{X}$, with order defined by $x \leq y$ iff $p \leq q + z$ where $x = \gamma(p)$, $y = \gamma(q)$ and $z \in \mu(0)$.

(ii) If X is an Abstract p-space $1 \leq p < \infty$ then so is $\hat{X}$.

(iii) Let X, Y be B-spaces, $T: X \to Y$ be a bounded linear operator. Then the extension $T: {}^*X \to {}^*Y$ induces a bounded linear operation $\hat{T}: \hat{X} \to \hat{Y}$ with $\|\hat{T}\| = \|T\|$. Actually if $T: {}^*X \to {}^*Y$ is an internal linear operator with $\|T\|$ being a finite hyper real number, then T induces a bounded linear operator $\hat{T}$ from $\hat{X}$ into $\hat{Y}$ such that $\|\hat{T}\| = \mathrm{st}\ (\|T\|)$.

(iv) If, in particular, Y is $\mathbb{C}$, then we see that $(X^*)^\wedge$ is isometrically imbedded in $(\hat{X})^*$.

(v) Let $T: X \to Y$ be a bounded linear operator. Then $\hat{T}$ is bijective, isometric if $\hat{T}$ is respectively so. If X and Y are Banach lattices, then $\hat{T}$ is order preserving or lattice isomorphism if T is respectively so.

Section 2.2:

Let A be a normed algebra. By transfer principle the norm function defined on $^*A$ will also satisfy the following condition.

$$* \models (\forall x \in {}^*A)\ (\forall y \in {}^*A)\ (\|xy\| \leq \|x\| \cdot \|y\|).$$

Observe that fin $(^*A)$ is a subalgebra and $\mu(0)$, the monad at 0 is an ideal in fin $(^*A)$. Therefore, $\hat{A}$ is a normed algebra over $\mathbb{C}$. As $^*\mathcal{M}$ is $\mathcal{K}$-saturated, $\hat{A}$ is a Banach algebra if A is so.

2.2.1 Proposition:

Let A be a B Algebra and $\hat{A}$ be its non-standard hull. Then the following are true.

(i) If A has a unit then $\hat{A}$ will also have unit.

(ii) If A is commutative then $\hat{A}$ is also commutative.

(iii) If A is a $*$ algebra then $\hat{A}$ is also a $*$ algebra.

(iv) If A is a $C*$ algebra then $\hat{A}$ is also a $C*$ algebra.

Proof:

(i) If A has a unit e then

$$\models (\exists e \varepsilon A)(\|e\| = 1)(\forall\, x \varepsilon A)(xe = x = ex).$$

By transfer principle,

$$* \models (\exists e \varepsilon {}^*A)(\|e\| = 1)(\forall\, x \varepsilon {}^*A)(xe = x = ex).$$

We observe that this e is actually in $\hat{A}$ and is a unit for $\hat{A}$.

(ii) We apply transfer principle to the extension of multiplication function.

$$\models (\forall\, x \varepsilon A)(\forall\, y \varepsilon A)(xy = yx)$$

$$* \models (\forall\, x \varepsilon {}^*A)(y \varepsilon {}^*A)(xy = yx).$$

So ${}^*A$ is commutative, and therfore $\hat{A}$ is also commutative.

(iii) Since A is a $*$ algebra there exists an involution $*: A \to A$ such that $\|x^*\| = \|x\|$. Applying the transfer principle again, we see that $*$ is an involution on ${}^*A$ and it induces an involution on $\hat{A}$ so that $\hat{A}$ becomes a $*$ algebra.

(iv) Let A be a $C*$ algebra. Let $\gamma(p) = x \varepsilon \hat{A}$, $p \varepsilon \text{ fin } ({}^*A)$. Then

$$\|x^*x\| = st(\|p^*p\|) = st(\|p^*\|\,\|p\|) = (st\|p^*\|)(st(\|p\|))$$
$$= \|x^*\|\,\|x\| = \|x\|^2 \text{ by (iii)}.$$

Hence the proposition.

2.2.2 Proposition:

Let A be a B-algebra with bounded right approximate identity $\{e_i\}_{i\varepsilon I}$. Then, the following hold.

(i) There exists $e\varepsilon\hat{A}$ such that $xe = x \;\forall\; x\varepsilon A$.

(ii) If we further assume that $\|e_i\| \leq 1 \;\forall\; i\varepsilon I$, then there exists a isometric isomorphism from the algebra of left multipliers $M_{\ell}(A)$ into $\hat{A}$.

Proof:

Let $M > 0$ be such that $\|e_i\| \leq M \;\;\forall\; i\varepsilon I$. By transfer principle there exists a net $\{e_i\}_{i\varepsilon {}^*I}$ in ${}^*A$ such that $\|e_i\| \leq M \;\;\forall\; i\varepsilon {}^*I$. By concurrence theorem there exists a $k \;\varepsilon\; {}^*I$ such that $k \geq i \;\;\forall\; i\varepsilon I$ (Refer Robinson [37, p.97]). Let $x\varepsilon A$ and $\varepsilon > 0$. Then there exists i_o such that

$\models (\forall\; i\varepsilon I)((i>i_o) \Rightarrow \|xe_i - x\| < \varepsilon)$. Therefore,

$* \models (\forall\; i\varepsilon {}^*I)((i>i_o) \Rightarrow \|xe_i - x\| < \varepsilon)$.

Since $k \geq i \;\forall\; i\varepsilon I$, $\|xe_k - x\| < \varepsilon \;\;\forall\; x\varepsilon A$, and $\forall\; \varepsilon>0$, therefore,

$$xe_k \approx x \quad \forall\; x\varepsilon A. \tag{1}$$

Since $\|e_k\| \leq M$, $e_k \;\varepsilon$ fin $({}^*A)$. Again e_k will not belong to $\mu(0)$ for, if so, then (1) implies that $x \;\varepsilon\; \mu(0) \;\forall\; x\varepsilon A$ which is absurd. Therefore, $\gamma(e_k) \neq 0$

Let e denote $\gamma(e_k)$. Then we observe that,

$$\|e\| \leq M \text{ and } x = \gamma(x) = \gamma(x.e_k) \text{ by (1)}$$
$$= \gamma(x)\,\gamma(e_k) = xe.$$

We prove (ii). Let $T \in M_{\ell}(A)$. Then $T: A \to A$ is a bounded linear operator such that $T(ax) = aT(x)$ $\forall$ $a, x \in A$. Now T induces a bounded linear operator $\hat{T}$ from $\hat{A}$ into $\hat{A}$ and by transfer principle $T(ax) = aT(x)$ $\forall$ $a, x \in A$.

Define a map $F: M_{\ell}(A) \to A$ by $F(T) = \hat{T}(e)$.

Then F is linear. If $T = \lambda(x)$, then $F(T) = x$.

For $x \in A$, $T(x) = \hat{T}(xe) = x\hat{T}(e)$ (2)

We prove that $\| F(T) \| = \|T\|$ $\forall$ $T \in M_{\ell}(A)$.

$$\|F(T)\| = \| \hat{T}(e) \| \leq \|\hat{T}\| \; \|e\| = \|T\| \; \|e\| \leq \|T\|.$$

by proposition 2.1.1(iii).

Therefore, $\|F(T)\| \leq \|T\|$.

Conversely, $$\|T\| = \sup_{\|x\| \leq 1} \frac{\|T(x)\|}{\|x\|} = \sup_{\|x\| \leq 1} \frac{\|x\hat{T}(e)\|}{\|x\|} \text{ by (2)}$$

$$\leq \|\hat{T}(e)\| = \|F(T)\|.$$

Therefore $\|F(T)\| = \|T\|$.

Hence the theorem.

<u>2.2.3</u> Theorem:

Let A be a commutative B-algebra with bounded approximate identity $\{e_i\}_{i \in I}$. Let $\Delta(A)$ denote its maximal ideal space. Then the following hold.

(i) $^*(\Delta(A))$ is contained in $\Delta(\hat{A})$.

(ii) Inclusion map from $^*\Delta(A)$ into $\Delta(\hat{A})$ is continuous.

(iii) If A is further assumed to be M-regular for some $M > 0$ then the Q-topology in $^*\Delta(A)$ is the same as relative topology of $^*(\Delta(A))$ in $\Delta(\hat{A})$.

Remark:

For the definition of M-regular B-algebras refer 1.2.2. We discussed Q-topology in Section 1.3.

Proof:

We denote $\Delta(A)$ by S throughout the proof for the sake of simplicity. Now,

$$\models (\forall \ \omega \varepsilon S)(\forall \ x,y \varepsilon A)(\omega(xy) = \omega(x)\,\omega(y)).$$

By transfer principle,

$$* \models (\forall \ \omega \ \varepsilon \ {}^{*}S)(\forall \ x,y \varepsilon {}^{*}A)(\omega(xy) = \omega(x)\,\omega(y)).$$

Again,

$\models (\forall \ \omega \varepsilon S)(\forall \ x \varepsilon A)(|\omega(x)| \leq \|x\|)$. Therefore,

$$* \models (\forall \ \omega \varepsilon {}^{*}S)(\forall \ x \varepsilon {}^{*}A)(|\omega(x)| \leq \|x\|).$$

Hence ω takes finite elements into finite elements, and $\|\omega\| \leq 1$.

Let $K > 0$ be such that $\|e_i\| \leq K \ \forall \ i \varepsilon I$. We know that $\forall \ \omega \varepsilon S$, $\lim\limits_{i} \omega(e_i) = 1$. Therefore the following is true.

$$\models (\forall \ \omega \varepsilon S)(\exists \ x \varepsilon A \wedge \|x\| \leq K) \ (1/2 < |\omega(x)|).$$

By transfer principle again,

$$* \models (\forall \ \omega \ \varepsilon \ {}^{*}S)(\exists \ x \varepsilon \ {}^{*}A \wedge \|x\| \leq K) \ (1/2 < |\omega(x)|).$$

Therefore, $\forall \ \omega \ \varepsilon \ {}^{*}S, \ \omega \not\approx 0$.

Hence ω induces a non-zero complex homomorphism on A.

Therefore, ${}^{*}S \subseteq \Delta(A)$.

We prove (ii). Notice that $\forall \ \tau \varepsilon S$, the sets of the form

$$W_{\tau} \ (x_1, \ldots, x_n : \varepsilon) = \{\omega \varepsilon S : |\omega(x_i) - \tau(x_i)| < \varepsilon ; 1 \leq i \leq n\},$$

$x_1, \ldots, x_n$ in A, $\varepsilon > 0$; form basic open neighbourhood system at τ, with respect to the Gelfand topology defined on S.

The Q-topology defined on $^*S$ will contain the following internal open sets.

$$\{\omega \varepsilon {}^*S\colon\ |\tau(p_i) - \omega(p_i)| < \varepsilon\ ,\quad 1 \leq i \leq n\}$$

where $p_1, \ldots, p_n$ is any finite collection of elements in $^*A$ and ε is any positive hyper real number.

Now we proceed to give a proof for (ii).

Let $\tau_o \varepsilon {}^*S; \varepsilon > 0$ be a standard real number. $x \varepsilon A$. Let $p \varepsilon$ fin $*(A)$ such that $\gamma(p) = x$.

Consider $W = \{\tau \varepsilon {}^*S\colon\ |\tau(x) - \tau_o(x)| < \varepsilon\}$.

Let $V = \{\tau \varepsilon {}^*S\colon\ |\tau(p) - \tau_o(p)| < \varepsilon\}$. Then V is an internal open subset containing τ_o and $V \leq W$ for, if $\tau \varepsilon V$ then,

$$|\tau(x) - \tau_o(x)| = st(|\tau(p) - \tau_o(p)|) < \varepsilon.$$

Hence Q-topology is finer than the Gelfand topology on S.

We prove (iii). Since A is M-regular the following is true.

$$\models (\forall\ \tau \varepsilon S)(\forall\ U \varepsilon \mathfrak{J} \wedge \tau \varepsilon U)((\exists p \varepsilon A \wedge \|p\| \leq M)\ (\hat{p}/U^c = 0 \wedge |\tau(p)| > 1/2)).$$

By transfer principle,

$$* \models (\forall\ \tau \varepsilon {}^*S)(\forall\ U \varepsilon {}^*\mathfrak{J} \wedge \tau \varepsilon U)((\exists p \varepsilon {}^*A \wedge \|p\| \leq M)$$
$$(\hat{p}/U^c = 0 \wedge |\tau(p)| > 1/2)).$$

Notice that $p \not\approx 0$ for, $|\tau(p)| > 1/2$.

Let $\tau \varepsilon {}^*S$. Let W be Q-open. Then there exists an internal open subset V such that $\tau_o \varepsilon V \subseteq W$.

Let $p \varepsilon {}^*A$ be as above with respect to τ_o and V. Take $x = \gamma(p)$.

Then, if $U = \{\tau \varepsilon {}^*S\colon\ |\tau(x) - \tau_o(x)| < 1/2\}$, we have $\tau_o \varepsilon U \subseteq V$.

Hence the relative topology of $^*S$ is finer than the Q-topology.

Hence the theorem.

2.2.4 Theorem:

Let A be a commutative M-regular semisimple B-algebra with bounded approximate identity. Moreover, we assume that $\hat{A}$ is also a regular B-algebra. Then $^*(\Delta(A))$ is dense in $\Delta(\hat{A})$.

Proof:

Let $S = \overline{^*(\Delta(A))}$. Suppose that $S \neq \Delta(\hat{A})$.
Then, by regularity of $\hat{A}$, there exists a $x = \gamma(p)$; $p \varepsilon \operatorname{fin}(^*A)$ such that $x \neq 0$ and $\hat{x}/S = 0$.
i.e., $\forall\, \omega \varepsilon {}^*\Delta(A)$, $\omega(p) \approx 0$. Then the following holds.

$$* \models (p \varepsilon {}^*A \wedge p \neq 0)(\forall\, \varepsilon > 0)(\forall \omega\, \varepsilon {}^*\Delta(A))(|\omega(p)| < \varepsilon).$$

By transfer principle,

$$\models (p \varepsilon A \wedge p \neq 0)\ (\forall\, \varepsilon > 0)(\forall \omega\, \varepsilon \Delta(A))(|\omega(p)| < \varepsilon).$$

i.e. there exists a $p \neq 0$ and $\omega(p) = 0\ \forall \omega\, \varepsilon \Delta(A)$ which contradicts the fact that A is semisimple.

Therefore, $^*\Delta(A)$ is dense in $\Delta(\hat{A})$.

Hence the theorem.

2.2.5 Corollary:

Let S be a locally compact T_2 space. Then there exists another locally compact T_2 space $\hat{S}$ such that

(i) $C_o(S)^\wedge$ is isometrically isomorphic to $C_o(\hat{S})$.

(ii) $^*S$ is topologically imbedded as a dense subset, in $\hat{S}$.

Proof:

Observe that $C_o(S)$ is a commutative C*-algebra. Therefore $C_o(S)^\wedge$ is again a commutative C*-algebra by proposition 2.2.1.

So, $C_o(S)^\wedge \cong C_o(\hat{S})$ for some locally compact T_2-space S. Moreover observe that $C_o(S)^\wedge$ is regular. Hence theorem 2.2.4 implies the result.

2.2.6 Remarks:

(i) In the above theorem, if we suppose that S is compact, then $\hat{S}$ will again be compact space.

It follows by proposition 2.2.1 and the fact that the only commutative C*-algebras with units are of the form C(K) where K is a compact T_2-space.

(ii) The above corollary for a compact T_2-space was proved by Henson. (See proposition 3.2 of Henson [25]).

Section 2.3:

Let A be a B-algebra; (π, X), a bounded Banach space representation. In this section we construct a new B-space representation out of (π, X) using non-standard methods. We shall be calling it as a non-standard hull of (π, X) which is going to be a useful tool for our studies on B-space representations.

Applying the transfer principle to (π, X), we have, a linear map $x \to {}^*\pi(x)$ from $^*A$ into $L({}^*X)$ such that

$$||{}^*\pi(x)|| \leq ||x|| \text{ and } {}^*\pi(xy) = {}^*\pi(x) \circ {}^*\pi(y).$$

(Let us recall that $||\pi(x)|| \leq ||x|| \ \forall \ x \in A$).

Therefore $\forall \ x \in A$, $||{}^*\pi(x)||$ is a finite hyper real number and so by proposition 2.1.1 (iii), $^*\pi(x)$ induces a linear operator on $\hat{X}$ and if we denote the induced operator by $\hat{\pi}(x)$, it satisfies the following:

$\forall\ x,y\varepsilon A,\quad \hat{\pi}(xy) = \hat{\pi}(x)\circ\hat{\pi}(y);$

$\|\hat{\pi}(x)\| = st\|^{*}\pi(x)\| = \|\pi(x)\|$ by proposition 2.1.1 (iii) and therefore $\|\hat{\pi}(x)\| \leq \|x\|$.

Thus $(\hat{\pi},\hat{X})$ is a B-space representation of A and

$\|\hat{\pi}(x)\| = \|\pi(x)\|\ \forall\ x\varepsilon A.$

This is called non-standard hull of (π,X).

2.3.1 Remark:

In this paragraph we define what is known as non-degenerate non-standard hull of a representation (π,X). Since, non-standard hull of (π,X) discussed above need not be non-degenerate, we take out the essential part of it. i.e.,

Let $\tilde{X} = [\hat{\pi}(A)\hat{X}]^{-}$. Then $\tilde{X}$ is invariant under A.

Let $\tilde{\pi} = \hat{\pi}/\tilde{X}$. Then the following assertions hold.

(i) $(\tilde{\pi},\tilde{X})$ is a non-degenerate B-space representation of A and is called non-degenerate non-standard hull of (π,X).

(ii) X is isometrically imbedded in $\tilde{X}$ for,

$[\pi(A)X]$ is isometrically imbedded in $[\hat{\pi}(A)\hat{X}]$ and π is non-degenerate.

2.3.2 Remark:

Let G be a lc group. Let (π,X) be a B-space representation of G. As per our convention $\|\pi(t)\| = 1\ \forall\ t\varepsilon G$.

We define non-standard hull of (π,X) in the following way.

Now (π,X) can be extended to a non-degenerate B-space representation of $L^1(G)$, which as usual will be denoted by the

same notation (π,X). Let $(\tilde{\pi},\tilde{X})$ denote the non-degenerate non-standard hull of (π,X). This is going to be a representation of G which we call as non-standard hull of (π,X).

2.3.3 Remark:

For any representation, (π,X) not necessarily uniformly bounded, Wolff [42] defined a non-standard hull in the following way.

Consider $X^{o} = \{\xi \in \mathrm{fin}({}^{*}X) : \pi(t)\,\xi \approx \xi \quad \text{if } t \approx e\}$ (1)

then $\mu(0) \subseteq X^{o}$ and $\forall\, t \in G$, $\pi(t)$ leaves $\mu(0)$ invariant. Take $X_G = \dfrac{X^{o}}{\mu(0)}$ and for every t, take $\pi^{o}(t) = \pi(t)/X_G$. Then, (π^{o},X_G) is defined to be non-standard hull. (For more details we refer to the above mentioned paper).

But with our convention that $\|\pi(t)\| = 1\ \forall\, t \in G$, both the definitions are same. In order to conclude this we essentially need to show that $X_G = \tilde{X}$.

For every $\xi \in \mathrm{fin}\,({}^{*}X)$ and $\forall\, f \in L^{1}(G)$, ${}^{*}\pi(f)\,\xi \in X^{o}$.
Therefore $\hat{\pi}(f)\,\xi \in X_G\ \forall\, \xi \in X$.

Since X_G is closed and $\tilde{X} = [\hat{\pi}(A)\hat{X}]$ we have $\tilde{X} \subseteq X_G$.

Now we show that $X_G \subseteq \tilde{X}$.
Let $\xi \in X_G$, then $\xi \in X$. By (1) we have $\lim\limits_{i} \hat{\pi}(e_i)\,\xi = \xi$ where $\{e_i\}$ is a bounded approximate identity of $L^{1}(G)$.
Therefore, $\hat{\pi}(e_i)\,\xi \in [\hat{\pi}(A)X]$ and hence $\xi \in [\hat{\pi}(A)\hat{X}]^{-} = \tilde{X}$.

Hence $X_G = \tilde{X}$.

CHAPTER 3

B-SPACE REPRESENTATIONS OF A COMMUTATIVE BANACH ALGEBRA

This chapter deals with B-space representations (π, X) of a commutative B-algebra A. Domar and Lindahl [13] had already made some efforts to understand the nature of (π, X) by defining three different spectral notions for (π, X). Around the same time, W. Arveson and others defined the notion of spectral subspaces associated with a B-space representation of lca group. A good source for this is Combes and Delaroche [6, part I]. Now, in this chapter we pursue the investigations of Domar and Lindahl, by defining the spectral subspaces associated with a B-space representation of a commutative B-algebra and study them systematically.

In Section 3.1 we define the spectrum of π and the spectral subspaces after fixing certain notations. In Sec. 3.2 we define the spectrum of an element in the representation space and discuss the relation between spectrum of an element and the spectrum of π. Then we establish various equivalent characterizations of the spectrum of π. We discuss the relation between the spectrum of π and the spectrum of an operator $\pi(x)$, $x \in A$.

In Sec. 3.3, we give a necessary and sufficient condition for an element $0 \neq \xi$ in X to be a common eigenvector for $\pi(x)$: x ranges over A; in terms of the spectral subspaces. We prove some interesting corollaries of the above results. In Sec. 3.4 we give various examples illustrating our results. In Sec. 3.5 we decompose (π, X) into its minimal sub-representations, by assuming the spectrum of π is discrete. We derive some corollaries of this theorem. Finally, we give a relationship between the spectrum of a representation (π, X) and the spectrum of its non-standard hull.

Section 3.1:

In this section we fix up some notations exclusively meant for this chapter, define the 'spectral subspaces' associated with a representation and prove some of their elementary properties.

3.1.1 Notations and Assumptions:

Let A denote a commutative Banach algebra. (π, X) be a B-space representation. Throughout this chapter, we assume that (π, X) is non-degenerate. Since we want to use the machinery of Gelfand transform effectively, we need A to be regular and semisimple. Moreover, we want the basic example of a representation namely regular representation to be a part of our study. i.e., we want the regular representation to be non-degenerate and so we assume that A to be Tauberian. For technical reasons we also require that every singleton in $\Delta(A)$ is a set of spectral synthesis. In short, we assume that A is commutative regular

N-algebra (Cf. 1.2.3) throughout this chapter, unless we say explicitly otherwise.

The following notations are fixed.

$I_\pi = \{x\varepsilon A: \pi(x) = 0\}$, $I_{\pi,\xi} = \{x\varepsilon A: \pi(x)\xi = 0\}$; $\xi\varepsilon X$

Let U be any open subset of $\Delta(A)$. Then $R^\pi(U)$ denotes the weak-* closure of $\{\pi(x)\xi : x\varepsilon C$ and $\mathrm{supp}(\hat{x}) \subseteq U,\ \xi\varepsilon X\}$. We recall that by $x\varepsilon C$ we mean supp $\hat{x}$ is compact.

Let E be a closed subset of $\Delta(A)$. Then

$M^\pi(E) = \{R^{\bar{\pi}}(E^c)\}^\perp$ i.e.,

$= \{\xi\varepsilon X: \pi(x)\xi = 0$ for every $x\varepsilon C$ such that $(\mathrm{supp}\ \hat{x}) \cap E = \emptyset\}$.

(Recall that $\bar{\pi}$ denotes the contragradient representation of π).

<u>3.1.2</u> Definitions:

1. For every closed set E in $\Delta(A)$, $M^\pi(E)$ is called as a spectral subspace associated to π.

2. For every element ξ in X, spectrum of ξ with respect to π is defined to be the hull of the closed ideal $I_{\pi,\xi}$ and we denote this closed set in $\Delta(A)$ by $Sp_\pi\,\xi$.

Henceforth we will not put π in all the above notations, unless we need to specify the representation with which we are working.

<u>3.1.3</u> Definition:

The spectrum of π is defined to be the hull of the closed ideal I_π and is denoted by Sp π.

Before giving some examples, let us derive some elementary properties of spectral subspaces.

3.1.4 Proposition:

Let E be a closed set in $\Delta(A)$. Then,

$$M(E) = \{\xi\varepsilon X:\ \pi(x)\xi = 0 \quad \forall\ x\varepsilon J_o(E)\}$$

$$= \{\xi\varepsilon X:\ \mathrm{Sp}\ \xi \subseteq E\}.$$

Proof:

First equality follows from the fact that if $x\varepsilon A$ and supp $\hat{x}$ is compact then (supp $\hat{x}$) $\cap$ $E = \phi$ iff $\hat{x}$ vanishes on a neighbourhood of E.

Now we prove the second equality.

Let $\xi\varepsilon X$ be such that $\pi(x)\xi = 0\ \forall\ x\varepsilon J_o(E)$.
Then $J_o(E) \subseteq I_\xi$, so $h(I_\xi) \subseteq h(J_o(E))$.
Therefore $\mathrm{Sp}\ \xi \subseteq E$.
Conversely, let $\xi\varepsilon X$ be such that $\mathrm{Sp}\ \xi \subseteq E$. i.e., $h(I_\xi) \subseteq E$.
Therefore, $J_o(E) \subseteq I_\xi$ by 1.2.2.
Therefore $\forall\ x\varepsilon J_o(E)$, $\pi(x)\xi = 0$. Hence the equality follows.

3.1.5 Proposition:

Let U, U_i designate open sets and E, E_i designate closed subsets in $\Delta(A)$. Then the following hold.

(i) R(U) and M(E) are π- invariant subspaces of X,

(ii) If $U_1 \subseteq U_2$ then $R(U_1) \subseteq R(U_2)$,

If $E_1 \subseteq E_2$ then $M(E_1) \subseteq M(E_2)$.

(iii) If $U \subseteq E$, then $R(U) \subseteq M(E)$.

(iv) $R(\emptyset) = M(\emptyset) = 0$,

$R(\Delta(A)) = M(\Delta(A)) = X$.

(v) If $U = \bigcup_{i\varepsilon I} U_i$ then $[\sum_{i\varepsilon I} R(U_i)]^- = R(U)$

and dually if $E = \bigcap_{i\varepsilon I} E_i$ then, $M(E) = \bigcap_{i\varepsilon I} M(E_i)$,

for any collection $\{U_i\}$ and $\{E_i\}$.

Proof:

(i) to (iii) can be easily seen by using the definitions.

(iv) follows because, π is non-degenerate.

To prove (v) we need the following lemma:

3.1.6 Lemma:

Let $x \varepsilon A$ with $\hat{x}$ having compact support. If $\operatorname{supp} \hat{x} \subseteq \bigcup_{i=1}^{n} U_i$ where U_i are open subsets, then there exists, $y_1, y_2, \ldots, y_n$ in A such that $\forall\, i$, $\operatorname{supp} \hat{y}_i$ is compact and $\operatorname{supp} \hat{y}_i \subseteq U_i$ and $x = \sum_{i=1}^{n} y_i$.

Proof:

We prove the lemma for $n = 2$.

Let K denote the $\operatorname{supp} \hat{x}$. For every $\tau \varepsilon K \cap U_1$ there exists V_τ such that $\overline{V}_\tau$ is compact and $\tau \varepsilon V_\tau \subseteq \overline{V}_\tau \subseteq U_1$. Similarly for elements in $K \cap U_2$.

Using compactness of K, choose a finite collection of $\{V_\tau\}$, $\tau \varepsilon K\cap U_1$ and $\{W_\tau\}$, $\tau \varepsilon K\cap U_2$ say $V_1, V_2, \ldots, V_m$ and $W_1, W_2, \ldots, W_n$ satisfying the following.

$$K \subseteq \left(\bigcup_{i=1}^{m} V_i\right) \cup \left(\bigcup_{j=1}^{n} W_j\right) \text{ and } V_i \subseteq U_1;\ W_j \subseteq U_2.$$

Take $V = \bigcup_{i=1}^{m} V_i$ and $W = \bigcup_{j=1}^{n} W_j$.

We observe that $\bar{V}$ and $\bar{W}$ are compact.

Now, $K \subseteq U_1 \cup W \subseteq U_1 \cup \bar{W} \subseteq U_1 \cup U_2$. Use regularity to get a y in A such that

$$\hat{y}/\bar{W} = 1 \text{ and } \hat{y}/U_2^c = 0.$$

Let, $z = x-xy$. Then supp $\hat{z} \subseteq U_1$ and supp $(xy)^\wedge \subseteq U_2$. Moreover $x = xy+z$; $(xy)^\wedge$ and $\hat{z}$ will have compact support.

Hence the Lemma for $n = 2$ and by induction for any n.

Now, we complete the proof of (v) of Proposition 3.1.5.

It is clear that $(\sum_i R(U_i))^- \subseteq R(U)$.

Let us prove $R(U) \subseteq (\sum_i R(U_i))^-$.

Let $x \in C$ be such that supp $\hat{x} \subseteq \bigcup_i U_i$. Since supp $\hat{x}$ is compact, there exists a finite collection $U_1, \ldots, U_n$ such that supp $\hat{x} \subseteq \bigcup_{i=1}^{n} U_i$.

By the above lemma there exist $y_1, y_2, \ldots, y_n$ in C such that supp $\hat{y}_i \subseteq U_i$ and $x = \sum_{i=1}^{n} y_i$.

Therefore, $\pi(x)\xi = \sum_{i=1}^{n} \pi(y_i)\xi$ and hence it follows that $R(U) \subseteq (\sum_{i \in I} R(U_i))^-$.

Second assertion follows immediately from the definition of M(E) and by usual duality arguments, if we apply first assertion

Hence the proposition 3.1.5.

3.1.7 Remark:

By (iv) and (v) of the above proposition we infer that there exists a minimal closed set among all the closed sets E in $\Delta(A)$ such that $M(E) = X$. Denote it by Λ.

Section 3.2:

In this section, we shall give various equivalent conditions for an element τ to belong to $\mathrm{Sp}\,\pi$ involving spectral subspaces and spectrum of an element in X. We define approximate point spectrum of π and determine its relation with the spectrum of π. Finally, we shall prove the equivalence of various definitions of spectrum of π in Theorem 3.2.9.

3.2.1 Proposition:

Let (π, X) be a B-space representation of A. Then the following are equivalent.

(i) $\tau \in \mathrm{Sp}\,\pi$

(ii) $\forall$ open neighbourhood U of τ, $R(U) \neq 0$.

(iii) $\tau \in \Lambda$

Proof:

Let $\tau \in \mathrm{Sp}\,\pi$. If U is any neighbourhood of τ, then there exists $x \in A$ such that $\hat{x}(\tau) = 1$ and $\mathrm{supp}\,\hat{x} \subseteq U$.

Suppose that $R(U) = 0$. Then we have, $\pi(x)\xi = 0 \;\forall\; \xi \in X$. So $x \in I$, but $\tau(x) = 1$. This contradicts that $\tau \in \mathrm{Sp}\,\pi$.

Hence $R(U) \neq 0$.

We prove (ii) $\Rightarrow$ (iii). Let $\tau \in \Delta(A)$ satisfy the hypothesis. Suppose $\tau \notin \Lambda$. Then there exists U of τ such that $U \cap \Lambda = \emptyset$. Now if $x \in C$ and supp $\hat{x} \subseteq U$, then (supp $\hat{x}$) $\cap \Lambda = \emptyset$ and so, $\pi(x)(\xi) = 0 \quad \forall \quad \xi \in M(\Delta)$. But since, $M(\Lambda) = X$, we have $\pi(x) = 0$. Therefore $R(U) = 0$, which contradicts the hypothesis.

We prove (iii) $\Rightarrow$ (i). We show that $\Lambda \subseteq \mathrm{Sp}\ \pi$ by proving that $M(\mathrm{Sp}\ \pi) = X$. Since $I \subseteq I_\xi \quad \forall \quad \xi \in X$, $h(I_\xi) \subseteq h(I)$. So $\mathrm{Sp}\ (\xi) \subseteq \mathrm{Sp}\ \pi$. Therefore $M(\mathrm{Sp}\ \pi) = X$.

Hence the proposition.

<u>3.2.2</u> Corollary:

For every neighbourhood W of Sp π, we have $R(W) = X$.

<u>Proof</u>:

We prove that, $\forall \quad \xi \in X$, $x \in C$, $\pi(x)\xi \in R(W)$.
Let $x \in C$. Then supp $\hat{x}$ is compact and supp $\hat{x} \subseteq W \cup (\mathrm{Sp}\ \pi)^c$.
By lemma 3.1.6 there exist y_1, y_2, in C, such that,

$$x = y_1 + y_2 \text{ and supp } \hat{y}_1 \subseteq W; \text{ supp } \hat{y}_2 \subseteq (\mathrm{Sp}\ \pi)^c.$$

So, $\pi(y_2)\xi = 0. \quad \forall \quad \xi \in M(\mathrm{Sp}\ \pi) = X$.
Therefore, $\pi(x)\xi = \pi(y_1)\ \xi \in R(W)$. Since π is non-degenerate we conclude that $X \subseteq R(W)$.

Hence the result.

The next proposition discusses the relation between spectrum of π and spectrum of each element in X.

<u>3.2.3</u> Proposition:

Let (π, X) be a B-space representation of A. Then,

$\mathrm{Sp}\ \pi = [\bigcup_{\xi \varepsilon Y} \mathrm{Sp}\ \xi]^-$ holds for any total set Y in X.

Proof:

By proposition 3.1.4 $[\bigcup_{\xi \varepsilon Y} \mathrm{Sp}\ \xi]^-$ is contained in Sp π.

We prove the other inclusion.

Let E denote the set $[\bigcup_{\xi \varepsilon Y} \mathrm{Sp}\ \xi]^-$. Suppose that $\tau \notin E$.

Let U be a neighbourhood of τ such that $\bar{U} \cap E = \emptyset$ and $\bar{U}$ is compact.

Then we prove that, $R(U) = 0$.

If $x \varepsilon C$ is such that supp $\hat{x} \subseteq U$, then x vanishes on a neighbourhood of E. So, $\hat{x}$ vanishes on a neighbourhood of Sp ξ $\forall$ $\xi \varepsilon Y$.

i.e. $\forall$ $\xi \varepsilon Y$, $x \varepsilon J_o(\mathrm{Sp}\ \xi)$. But $J_o(\mathrm{Sp}\xi) \subseteq I_\xi$.

Therefore, $\pi(x)\xi = 0$ $\forall$ $\xi \varepsilon Y$. Since Y is total, $\pi(x) = 0$.

Thus, $R(U) = 0$.

By proposition 3.2.1 (ii) we conclude that $\tau \notin$ Sp π.

Hence the proposition 3.2.3.

3.2.4 Remark:

Let (π, X) be a B-space representation of A, $x \varepsilon A$, $\xi \varepsilon X$. Then, we have $\mathrm{Sp}\ (\pi(x)\xi) \subseteq (\mathrm{supp}\ \hat{x}) \cap \mathrm{Sp}\ \xi$.

It is evident that $\mathrm{Sp}\ (\pi(x)\xi) \subseteq \mathrm{Sp}\ \xi$.

We show that $\mathrm{Sp}\ (\pi(x)\xi) \subseteq \mathrm{Supp}\ \hat{x}$. If $\tau \notin$ supp $\hat{x}$, then there exist an open neighbourhood U of τ such that $\hat{x}/U = 0$.

Choose an element $a \varepsilon A$ such that $\hat{a}(\tau) = 1$ and supp $\hat{a} \subseteq U$.

Then $ax = 0$ and so, $a \varepsilon I_{\pi(x)\xi}$, but $\hat{a}(\tau) = 1$.

Therefore $\tau \notin \mathrm{Sp}\ (\pi(x)\xi)$.

3.2.5 Corollary:

Let (π, X) be a cyclic representation with ξ being cyclic vector. Then we have, $\operatorname{Sp} \pi = \operatorname{Sp} \xi$.

Since ξ is a cyclic vector, the subspace $\pi(A)\xi$ is total in X. By the above remark, we have

$\operatorname{Sp} \zeta \in \operatorname{Sp} \xi$. $\forall \zeta \in \pi(A)\xi$.

Therefore, $[\bigcup_{\zeta \in \pi(A)\xi} \operatorname{Sp} \zeta]^- \subseteq \operatorname{Sp} \xi$.

By proposition 3.2.3, $\operatorname{Sp} \pi \subseteq \operatorname{Sp} \xi \subseteq \operatorname{Sp} \pi$.

Hence the corollary.

Remarks:

The notion of spectrum of an element in X of a representation space of a locally compact abelian group is found in Domar [12]. But even earlier, the spectrum of f in $L^\infty(G)$ was studied, see for example, Beurling [2], Godement [19]. Refer Reiter [35, Chapter 7] for further details. Arveson [1] again defined the spectrum for Group representations and connected it with spectral subspaces associated with π.

When the group G is non abelian, Eymard [15] defined the spectrum of an element in $(A_2(G))^*$, where $A_2(G)$ is the B-algebra we defined in 1.2.3. The works of Eymard and Herz related to the above will be discussed in 3.4.9

As in the classical case, we say that a nonzero vector ξ in X is an eigenvector for π if ξ is simultaneously an eigenvector for all commuting family of operators $\pi(x)$; x ranging in A. i.e. There exists a complex valued function $a \to \alpha_a$ on A such that $\pi(a)\xi = \alpha_a \cdot \xi \ \forall \ a \varepsilon A$.

It is trivial to check that such a function is a complex homomorphism on A. It necessitates the following definition.

3.2.6 Definition:

An element ω of $\Delta(A)$ is said to belong to the Point Spectrum of π if there exists a nonzero ξ in X such that $\pi(a)\xi = \omega(a)\xi \ \forall \ a \varepsilon A$, and is denoted by P_π.

There is one more concept called Approximate Point Spectrum for a bounded linear operator T on X, X a B-space. We say that a complex number α is said to be in approximate point spectrum if there exists a net $\{\xi_i\}$ in X such that

$$\| \xi_i \| = 1 \quad \forall \ i \text{ and } \lim_i \| T(\xi_i) - \alpha \xi_i \| = 0.$$

The following definition generalizes the above, for the commuting family of operators $\{\pi(x) : x \varepsilon A\}$.

3.2.7 Definition:

An element $\omega \ \varepsilon \ \Delta(A)$ is said to belong to the Approximate Point Spectrum of π if there exists a net $\{\xi_i\}_{i \varepsilon I}$ in X such that

$$\| \xi_i \| = 1 \text{ and } \lim_i \| \pi(a)\xi_i - \omega(a)\xi_i \| = 0.$$

This set will be denoted by AP_π.

3.2.8 Remarks:

(i) Observe that $P_\pi \subseteq AP_\pi$.

(ii) The following theorem was proved by Domar and Lindahl [13, Theorem 6.5].

Theorem:

Let A be a regular semisimple B-algebra, (π, X) a B-space representation. Then, $\mathrm{Sp}\,\pi = AP_\pi$.

In the following theorem we summarize various characterizations we obtained earlier for $\mathrm{Sp}\,\pi$ in view of the above theorem.

3.2.9 Theorem:

Let A be a commutative regular N algebra and (π, X) a B-space representation. For $\tau \in \Delta(A)$, the following are equivalent.

(i) $\tau \in \mathrm{Sp}\,\pi$

(ii) For every open neighbourhood U of τ, $R(U) \neq 0$.

(iii) $\tau \in \Lambda$

(iv) $\tau \in AP_\pi$.

(v) $\tau \in [\bigcup_{\xi \in Y} \mathrm{Sp}\,\xi]^-$, where Y is any total set in X.

(vi) For every x in A, $|\tau(x)| \leq \|\pi(x)\|$

(vii) $\tau \in \Delta(B)$ where B is the Banach algebra generated by $\pi(A)$ in $L(X)$.

Proof:

In view of propositions 3.2.1, 3.2.3 and the theorem cited in Remark 3.2.8, we need to show only that (iv) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i). But (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i) is trivial to prove.

Now, we prove (iv) $\Rightarrow$ (vi).

Suppose $\tau \varepsilon AP_\pi$. Then there exists a net $\{\xi_i\}$ in X as in the definition 3.2.7.

For $x \varepsilon A$ we have,

$$|\tau(x)| = \|\tau(x)\ \xi_i\|$$

$$\leq \|\pi(x)\ \xi_i - \tau(x)\ \xi_i\| + \|\pi(x)\ \xi_i\|$$

Taking the limits we get that,

$$|\tau(x)| \leq \liminf_i \|\pi(x)\xi_i\|$$

$$\leq \|\pi(x)\|$$ as we required.

Hence the theorem.

Remark:

The above theorem was proved for the group algebras by A. Connes. (See Combes and Delaroche [6]).

We list some of the corollaries of the above theorem.

3.2.10 Corollary:

Let A, (π, X), B be as in theorem 3.2.9. Then Sp π and $\Delta(B)$ are homeomorphic.

By the theorem 3.2.9 any $\tau \varepsilon$ Sp π defines an element in $\Delta(B)$ and this correspondence is bijective. Since the Gelfand topology on $\Delta(B)$ is the weak-$*$ topology defined by B and hence by the norm dense subspace $\pi(A)$ in B, and π is a bounded representation, we evidently see that Sp π and $\Delta(B)$ are homeomorphic.

3.2.11 Corollary:

Sp π is compact iff the identity operator I_X belongs to B. It follows because $\Delta(B)$ is compact iff B has a unit.

3.2.12 Corollary:

Sp π = Sp $\bar{\pi}$, where $(\bar{\pi}, X^*)$ is the contragradient representation of (π, X).

It follows by (vi) of theorem 3.2.9.

The following theorem, which is again a corollary of theorem 3.2.9, shall perhaps justify the terminology of the spectrum of a representation.

We recall that the set $\{\alpha \in \mathbb{C}: (\alpha - T) \text{ is not invertible}\}$ is called as the spectrum of a bounded linear operator T on X, and is denoted by $\sigma(T)$.

3.2.13 Theorem:

Let (π, X) be a B-space representation of A. Then for every $x \in A$, $\sigma(\pi(x)) = (\hat{x}(\text{Sp } \pi))^-$ holds.

Proof:

Let $x \in A$. Then $(\hat{x}(\text{Sp } \pi))^- \subseteq \sigma(\pi(x))$ for, if $\omega \in \text{Sp } \pi = AP_\pi$, then there exists a net $\{\xi_i\} \subseteq X$ such that

$$\|\xi_i\| = 1 \text{ and } \lim_i \|\pi(x)\xi_i - \omega(x)\xi_i\| = 0.$$

Therefore, $\omega(x)$ belongs to the approximate point spectrum of $\pi(x)$ and hence $\omega(x)$ belongs to $\sigma(\pi(x))$.

We prove the other way.

Case (i):

Assume that Sp π is compact. Therefore B (as in theorem 3.2.9) contains the identity operator I.

$$\sigma(\pi(x)) \subseteq \sigma_B(\pi(x)) = \{\hat{x}(\omega): \omega \varepsilon \Delta(B) = \text{Sp } \pi\}$$

and hence the result.

Case (ii):

Assume that Sp π is not compact. Let B_1 be the B algebra after adjoining identity I.

$$\text{i.e.,} \quad B_1 = \{\alpha I+T: \quad T\varepsilon B, \ \alpha\varepsilon\mathbb{C}\}$$

Then for any $x\varepsilon A$ $\sigma_B(\pi(x)) = \sigma_{B_1}(\pi(x))$. By case (i),

$$\sigma_{B_1}(\pi(x)) \subseteq \{\hat{x}(\omega): \omega \varepsilon \Delta(B_1) = \Delta(B) \cup \tau_\infty\}.$$

Therefore, if we prove that, $\hat{x}(\tau_\infty) \ \varepsilon \ \{\hat{x}(\omega):\omega\varepsilon\Delta_B\}^-$ we are done. Thus we need to prove $0 \ \varepsilon \ \{\hat{x}(\omega): \omega \varepsilon \Delta_B\}^-$.
Suppose not, there exists $\varepsilon>0$ such that $\forall \ \omega \ \varepsilon$ Sp π,

$|\hat{x}(\omega)| \geq \varepsilon$. Consider the open neighbourhood,

$$W = \{\tau \varepsilon \Delta(B_1): |\tau(x)-\tau_\infty(x)| < \varepsilon/2\} \text{ at } \tau_\infty.$$

Then we have $W \cap \Delta(B) = \emptyset$. But since $\Delta(B_1)$ is the one-point compactification of $\Delta(B)$, it contradicts the fact that $\Delta(B)$ is not compact.

Hence the theorem.

Section 3.3:

In this section we shall give a necessary and sufficient condition for any complex homomorphism to belong to the point spectrum of π in terms of spectral subspaces associated with π. As a consequence we prove that every non zero ξ in X will have non-empty spectrum andany irreducible B-space representation will be 1-dimensional.

3.3.1 Theorem:

Let (π, X) be a B-space representation of A. Let ω_o be in $\Delta(A)$. Then ω_o is in the point spectrum of π iff $M(\{\omega_o\}) \neq 0$. Infact we prove that any non zero ξ in X is an eigenvector for π with ω_o as the eigenvalue iff $\xi \in M(\{\omega_o\})$.

We need thefollowing lemma.

3.3.2 Lemma:

Let (π, X) B-space representation of A. Let ω be in $\Delta(A)$. Then for every neighbourhood U of ω, there exist a compact neighbourhood K of ω, contained in U and an element a in A satisfying the following:

(i) $\omega(a) = 1$.

(ii) $\text{supp } \hat{a} \subseteq U$.

(iii) $\pi(a)\xi = \xi \ \forall \ \xi \in M(K)$.

Proof of the Lemma:

Choose a closed neighbourhood K and V a open neighbourhood of ω such that,

$\omega \in K \subseteq V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact, and an element a in A

such that $\hat{a}/\overline{V} = 1$ and supp $\hat{a} \subseteq U$. (*)

Claim: $\pi(a)\xi = \xi \quad \forall\ \xi \in M(K)$.

For this, we prove that, $(\pi(a)\xi,\eta) = (\xi,\eta)\ \forall \eta \in X^*$. (†)

Since $X^* =$ weak-* closure of $R^{\bar{\pi}}(K^c) + R^{\bar{\pi}}(V)$ where $\bar{\pi}$ is the contragradient representation of π, [Cf. 3.1.5 (v)], it is sufficient to prove (†) for every $\eta \in R^{\bar{\pi}}(K^c)$ and for every $\eta \in R^{\bar{\pi}}(V)$.

Suppose that $\eta \in R^{\bar{\pi}}(K^c)$. Then $(\xi,\eta) = 0$ by definition of $M(K)$ and $(\pi(a)\xi,\eta) = 0$ for $M(K)$ is invariant under A.

Let $\eta \in R^{\bar{\pi}}(V)$ be of the form $\bar{\pi}(b)(\eta')$ for some $\eta' \in X^*$ and supp $\hat{b} \subseteq V$.

Then by (*), we have $ab = b$. Therefore,

$$(\pi(a)\xi,\eta) = (\pi(a)\xi,\ \bar{\pi}(b)\eta')$$

$$= (\xi,\ \bar{\pi}(ab)\eta') = (\xi,\ \bar{\pi}(b)\eta') = (\xi,\eta).$$

Hence the claim.

Now we give the proof of Theorem 3.3.1.

Suppose that ω_o is in P_π, and $0 \neq \xi$ is an eigenvector corresponding to ω_o. Then,

$$\pi(a)\xi = \omega_o(a)\xi \quad \forall\ a\in A.$$

If $a \in A$ is such that supp $\hat{a}$ does not contain ω_o then,

$$\omega_o(a) = 0 \text{ implies } \pi(a)\xi = 0.$$

Therefore, $\xi \in M(\{\omega_o\})$.

Now we prove the converse.

Suppose that ξ is a non zero vector in $M(\{\omega_o\})$.
To prove ξ is an eigenvector, it is sufficient to prove that

$$\forall\ a\varepsilon C,\quad \pi(a)\xi = \omega_o(a)\xi.$$

Case (i):

Suppose that $\omega_o(a) = 0$. Let $\varepsilon>0$.
Since the set $\{\omega_o\}$ is a S-set we have, a neighbourhood U of ω_o and $b\varepsilon A$ such that $\hat{b}/U = 0$ and $\|a-b\| < \varepsilon$. So, $\omega_o \notin \operatorname{supp} \hat{b}$.
Since, $\xi \varepsilon M(\{\omega_o\})$, $\pi(b)\xi = 0$. Therefore,

$$\|\pi(a)\xi\| = \|\pi(a-b)\xi\| \leq \|\pi\|\ \|a-b\|\ \|\xi\| < \varepsilon\,\|\pi\|\ \|\xi\|.$$

Thus we have, $\pi(a)\xi = 0 = \omega_o(a)\xi$.

Case (ii):

Suppose that $\omega_o(a) \neq 0$.
Let K denote the compact set supp $\hat{a}$. Choose a $b\varepsilon A$ such that $\hat{b}/K = \omega_o(a)$. Then $\omega_o(a-b) = 0$. Therefore $\pi(a)\xi = \pi(b)\xi$ by Case (i)
Since $\omega_o \varepsilon \operatorname{Int}(K)$, there exists a neighbourhood W of ω_o such that $W \subseteq K$.

By lemma 3.3.2 we get an element x in A and a compact neighbourhood K_1 such that

$$\hat{x}(\omega_o) = 1 \quad \operatorname{supp} \hat{x} \subseteq W \text{ and } \pi(x)\xi = \xi \ \forall\ \xi \varepsilon M(K_1).$$

In particular, we have $\pi(x)\xi = \xi$. Observe that $bx = \omega_o(a)x$.
Therefore, $\pi(a)\xi = \pi(b)\xi = \pi(bx)\xi = \omega_o(a)\pi(x)\xi = \omega_o(a)\xi$

Hence the theorem.

Remark:

Theorem 3.3.1 is already known for $L^1(G)$. Refer Combes and Delaroche [6, proposition 3.5].

3.3.3 Corollary:

Let (π,X) be as in theorem 3.3.1. Then for every $\eta \in X^*$, $\eta \in M^{\bar{\pi}}(\{\omega\})$ iff $\bar{\pi}(a)\eta = \omega(a)\eta \quad \forall\, a \in A$.

3.3.4 Corollary: (Schur's theorem)

Let (π,X) be a B-space representation of A. Then π is topologically irreducible iff π is one-dimensional.

Proof:

If $X = 0$. Then there is nothing to prove. Assume that $X \neq 0$ so that $Sp\,\pi \neq \phi$.

Let $\omega \in Sp\,\pi$. Then by theorem 3.2.9, $R(W) \neq 0$ for every open neighbourhood W of ω. Since $R(W)$ is invariant and π is irreducible, we have $R(W) = X$.
By proposition 3.1.5 (iii), we have $M(\bar{W}) = X$. Hence by minimality of the spectrum, $Sp\,\pi \subseteq \bar{W}$ for every neighbourhood W of ω.
Therefore, $Sp\,\pi = \{\omega\}$. i.e., $M(\{\omega\}) = X$.
But then, by the above theorem, we have,

$$\forall\, \xi \in X, \quad \pi(a)\xi = \omega(a)\xi .$$

Thus X is 1-dimensional.

Hence the corollary.

<u>3.3.5</u> Corollary:

Let ξ be an element in X. Then $\xi = 0$ iff $Sp\ \xi = \phi$.

<u>Proof</u>:

If $\xi = 0$ then, $I_\xi = A$ so that $h(I_\xi) = \phi$ and hence $Sp(\xi) = \phi$.

Conversely, suppose that $Sp\ \xi = \phi$. i.e. $h(I_\xi) = \phi$. Since A is Tauberian, we have $I_\xi = A$ i.e. $\pi(x)\xi = 0\ \forall\ x \in A$. On the other hand, $Sp\ \xi = \phi$ implies $\xi \in M(\{\omega\})\ \forall\ \omega \in \Delta(A)$. Therefore, $0 = \pi(x)\xi = \omega(x)\xi\ \forall\ x \in A$ and $\forall\ \omega \in \Delta(A)$. Using the semisimplicity of A, we conclude that $\xi = 0$.

Hence the corollary.

<u>3.3.6</u> Corollary:

The set of all isolated points in $Sp\ \pi$ is contained in the point spectrum.

<u>Proof</u>:

Let $\omega \in Sp\ \pi$ be isolated. Choose a neighbourhood W of ω such that $W \cap Sp\ \pi = \{\omega\}$.

<u>Claim</u>: $R(W) \subseteq M(\{\omega\})$. (*)

Let $\pi(x)\xi \in R(W)$ with $supp\ \hat{x} \subseteq W$. Suppose that there is a $y \in A$ such that $\omega \notin supp\ \hat{y}$. Then

$$\begin{aligned} Sp\ (\pi(y).\pi(x)\xi) &\subseteq (supp\ \hat{y}) \cap (supp\ \hat{x}) \cap Sp\xi \text{ by Remark 3.2.4.} \\ &\subseteq (supp\ \hat{y}) \cap W \cap Sp\ \pi \\ &= \phi. \end{aligned}$$

By the above corollary, $\pi(y).(\pi(x)\xi) = 0$.

Therefore, $R(W) \subseteq M(\{\omega\})$.

But since, $\omega \varepsilon$ Sp π, we have $R(W) \neq 0$ and hence the corollary.

3.3.7 Remark:

We end this section by giving an example so that the assumption in Theorem 3.3.1 that every singleton in $\Delta(A)$ is a S-set is not superfluous.

Let $A = C^n[0,1]; n \geq 1$, the algebra of all complex valued n-times continuously differentiable functions on $[0,1]$, with the following norm,

$$||f|| = \sup_{0 \leq x \leq 1} \sum_{k=1}^{n} \frac{|f^k(x)|}{k!} \quad \forall \quad f \varepsilon A.$$

Let J denote the closed ideal $\{f \varepsilon A: f(0) = f'(0) = 0\}$

Set $X = A/J$. Define $\pi: A \to L(X)$ by $\pi(f)(\tilde{g}) = (fg)^{\sim}$; $\tilde{g}$ in X. Fix a $\tilde{h} \varepsilon X$ such that $h(0) \neq 0$. We denote the complex homomorphism $f \to f(t)$ by $\omega_t \ \forall \ t \varepsilon [0,1]$.

We prove that Sp $(\tilde{h}) = \{\omega_o\}$, but still $\tilde{h}$ is not an eigen vector for ω_o.

Trivially, $\omega_o \varepsilon \mathrm{Sp}(\tilde{h})$ and by taking an f in A such that $f(0) = f'(0) = 0$ but $f(t) \neq 0 \ \forall \ t \neq 0$, we conclude that $\omega_t \notin \mathrm{Sp}\ \{\tilde{h}\}$ if $t \neq 0$. Therefore $\mathrm{Sp}\ \{\tilde{h}\} = \{\omega_o\}$.

But if $\tilde{h}$ were an eigenvector for ω_o i.e.,

$$\pi(f)\ \tilde{h} = \omega_o(f)\ \tilde{h} \quad \forall \ f \varepsilon A$$

$$(fh)^{\sim} = f(0)\ \tilde{h}$$

Thus we have, $fh - f(0)h \ \varepsilon \ J \ \forall \ f$

i.e., $f'(0)\ h(0) = 0 \ \forall \ f \varepsilon A$

and so, $f'(0) = 0 \ \forall \ f \varepsilon A$ which is absurd.

Hence the result.

Section 3.4:

This section is devoted to the study of spectrum of an element in X, for a B-space representation (π,X) of A. We give necessary and sufficient conditions for an element τ in $\Delta(A)$ to be in the spectrum of an element in X. These conditions, are used to specify the spectrum of an element in particular cases.

The following theorem is quite useful in finding the spectrum of an element in X.

3.4.1 Theorem:

Let (π,X) be a B-space representation of A. Let $\xi \in X$; $\eta \in X^*$. Then we obtain the following.

(i) $\omega \in Sp_\pi(\xi)$ iff for every neighbourhood V of ω, there exists a $y \in A$ such that supp $\hat{y} \subseteq V$ and $\pi(y)\xi \neq 0$.

(ii) $\omega \in Sp_{\bar{\pi}}(\eta)$ iff for every neighbourhood V of ω, there exists $\xi \in X$ such that $Sp_\pi \xi \subseteq V$ and $(\xi,\eta) \neq 0$.

Proof:

(i) Let (β, Y) be sub-representation of π, with ξ as cyclic vector. Then $Sp\,\xi$ equals to $Sp\,\beta$ by Corollary 3.2.5. Therefore, $\omega \in Sp\,\xi$ iff for every neighbourhood V of ω, $R^\beta(V) \neq 0$(by Theorem 3.2.9.). i.e. there exists $\zeta \in Y$ and $y \in A$ such that supp $\hat{y} \subseteq V$ and $\beta(y)\zeta \neq 0$ i.e., $\pi(y)\zeta \neq 0$.

Since $[\pi(A)\xi]$ is norm dense in Y, there exists $x \in A$ such that, $\pi(y)\circ\pi(x)\,\xi \neq 0$ i.e. $\pi(xy)(\xi) \neq 0$.

Hence the assertion (i).

(ii) Let $\omega \in \Delta(A)$ satisfy the condition given in the hypothesis. Assuming $\omega(a) \neq 0$ for some $a \in A$, we prove that $\bar{\pi}(a)\eta \neq 0$.

Let V be a neighbourhood of ω such that $\tau(a) \neq 0 \quad \forall \tau \in V$. By the regularity of A, there exists a $b \in A$ such that $(ab)^{\wedge} = 1$ on some compact neighbourhood K of ω contained in V.

But by the proof of lemma 3.3.2, we infer that

$$\pi(ab)\xi = \xi \quad \forall \quad \xi \in M(K)$$

Now we invoke the condition given in the hypothesis to get a $\xi \in X$ such that $Sp\,\xi \subseteq K$, so that $\xi \in M(K)$ and $(\xi,\eta) \neq 0$. Therefore by the above $(\pi(ab)\xi, \eta) \neq 0$. Thus $\bar{\pi}(a)\eta \neq 0$ as we required. Hence $\omega \in Sp_{\bar{\pi}}(\eta)$.

Now we prove the converse. Suppose that ω does not satisfy the condition. Then there exists a neighbourhood V of ω such that for every $\xi \in X$ such that $Sp\,\xi \subseteq V$ implies $(\xi,\eta) = 0$ Let $a \in A$ be such that $\omega(a) = 1$ and supp $\hat{a} \subseteq V$.

$$\forall \quad \xi \in X, \quad Sp\,(\pi(a)\xi) \subseteq V \quad \text{and so,}$$

$$\begin{aligned} (\xi, \bar{\pi}(a)(\eta)) &= (\pi(a)\xi, \eta) \\ &= 0 \quad \forall \quad \xi \in X \quad \text{since } Sp\,(\pi(a)\xi) \subseteq V \quad \forall \quad \xi \in X. \end{aligned}$$

Therefore, $\bar{\pi}(a)\eta = 0$ but $\omega(a) = 1$. Thus $\omega \notin Sp_{\bar{\pi}}(\eta)$.

Hence the theorem.

In what follows we shall discuss some specific examples.

(1) The regular representation (λ, A).

3.4.2:

The following assertions hold for (λ, A).

(i) λ is non-degenerate.

(ii) $\mathrm{Sp}\,\lambda = \Delta(A)$.

(iii) $\mathrm{Sp}_\lambda(a) = \mathrm{supp}\,\hat{a}$ for every $a \in A$.

Proof:

Since A is regular and Tauberian, (i) follows.
(ii) is evident because A is semisimple.
We prove (iii), Let $\omega \in \mathrm{Sp}_\lambda(a)$.

If we apply the above theorem to λ, we see that $\omega \in \mathrm{Sp}_\lambda(a)$ iff for every neighbourhood V of ω there exists $b \in A$ such that $\mathrm{supp}\,\hat{b} \subseteq V$ and $ab \neq 0$,

iff for every neighbourhood V of ω $V \cap \{\tau \in \Delta(A): \hat{a}(\tau) \neq 0\} \neq \emptyset$ by regularity,

iff $\omega \in \mathrm{supp}\,\hat{a}$.

3.4.3:

For every closed set $E \subseteq \Delta(A)$, we have,

(i) $M^\lambda(E) = \{x \in A: \hat{x}(\tau) = 0 \ \forall \tau \notin E\}$.

(ii) If $M^\lambda(\{\omega\}) \neq 0$ then $M^\lambda(\{\omega\})$ is a 1-dimensional subspace.

(iii) $M^\lambda(\{\omega\}) \neq 0$ iff ω is isolated in $\Delta(A)$.

Proof:

Using the above result and Proposition 3.1.4, we see that

$$M^{\lambda}(E) = \{x\varepsilon A: \text{ supp } \hat{x} \subseteq E\}$$
$$= \{x\varepsilon A: \hat{x}(\tau) = 0 \ \forall \ \tau \notin E\}.$$

Hence the assertion (i).

(ii) By (i) and by semisimplicity of A, there exists an element $a_{\omega}\varepsilon A$ such that
$$\hat{a}_{\omega}(\tau) = 1 \text{ if } \tau = \omega$$
$$= 0 \text{ if } \tau \neq \omega$$

It is easy to check that $M^{\lambda}(\{\omega\})$ is spanned by a_{ω}.

(ii) yields (iii).

(2) The Contragradient Representation of (λ, A):

<u>3.4.4</u> Proposition:

As usual $(\bar{\lambda}, A^*)$ denotes the contragradient representation of (λ, A). Let $f\varepsilon A^*$ and $E_f = \{\bar{\lambda}(a)f: a\varepsilon A\}$. Then the following are equivalent.

(i) $\omega \ \varepsilon \ Sp_{\bar{\lambda}} \ (f)$

(ii) For every neighbourhood V of ω there exists a in A such that $\text{supp } \hat{a} \subseteq V$ and $f(a) \neq 0$.

(iii) ω belongs to the weak-* closure of E_f.

<u>Proof</u>:

By theorem 3.4.1, (i) and (ii) are equivalent.

We prove (i) and (iii) are equivalent. Suppose that $\omega \ \varepsilon \ Sp_{\bar{\lambda}} \ (f)$. Let $x\varepsilon A$ be such that $g(x) = 0 \ \forall \ g\varepsilon E_f$. Then $f(ax) = 0 \ \forall \ a\varepsilon \ A$ and so $\bar{\lambda}(x)f = 0$. Therefore $\omega(x) = 0$. Apply Hahn-Banach theorem to conclude that ω belongs to the weak-*closure of E_f.

By retracing the steps we see that converse also holds.

3.4.5 Corollary:

For every $\omega \varepsilon \Delta(A)$, $Sp_{\bar{\lambda}}\, \omega = \{\omega\}$.

Applying (iii) of the above proposition we get the result.

3.4.6 Proposition:

For every $\omega \varepsilon \Delta(A)$, $M^{\bar{\lambda}}(\{\omega\})$ equals the one-dimensional subspace spanned by ω in A^*.

Proof:

By theorem 3.3.1, any f in A^*, will belong to $M^{\bar{\lambda}}\{\omega\}$

iff $\quad \bar{\lambda}(a)f = \omega(a)f \quad \forall\ a\varepsilon A$ (1)

Hence $M^{\bar{\lambda}}\{\omega\}$ contains ω. If $f \varepsilon M^{\bar{\lambda}}\{\omega\}$, using (1) we can easily see that Ker $\omega \subseteq$ Ker f and therefore, $f = f(a)\omega$, where $\omega(a)=1$

Hence the proposition.

3.4.7 Corollary:

Let (π, Y) be a sub-representation of $\bar{\lambda}$. Suppose that Y is a nontrivial weak-$*$closed subspace. Then the following are true.

(i) $\quad Sp\ \pi = \Delta(A) \cap Y$.

(ii) $\quad$ If $\Delta(A) \cap Y$ is a finite set, then Y is finite dimensional.

Proof:

(i) $\quad$ Let $f\varepsilon Y$ and E_f be as defined in proposition 3.4.4.

Since Y is weak-$*$closed, E_f is contained in Y. Thus $Sp_{\pi}(f) \subseteq \Delta(A) \cap Y$

But $\forall\ \omega \varepsilon \Delta(A) \cap Y$, $Sp\,\omega = \{\omega\}$. Therefore, $Sp\ \pi \supseteq \Delta(A) \cap Y$ by theorem 3.2.9 (iv). Hence $Sp\ \pi = \Delta(A) \cap Y$.

(ii) Now we prove that Y is spanned by Sp π.

Let Sp $\pi = \{\omega_1, \ldots, \omega_n\}$. Choose $x_1, \ldots x_n$ in C such that $\omega_i(x_j) = \delta_{ij}$.

Let $f \varepsilon Y$. Then we claim that $f = \sum_{i=1}^{n} f(x_i)\, \omega_i$.

If $h = f - \sum_{i=1}^{n} f(x_i)\, \omega_i$, then we prove $h = 0$ by showing that $Sp_\pi\, h = \emptyset$.

$$\text{Since, } h \varepsilon Y, \quad Sp_\pi\, h \subseteq \{\omega_1, \ldots, \omega_n\} \tag{1}$$

Consider $\pi(x_k)\, f$.

$$Sp\ (\pi(x_k) f) \subseteq (\text{supp } \hat{x}_k) \cap Sp_\pi\ (f)$$
$$\subseteq \{\omega_k\}.$$

Therefore, by Proposition 3.4.6, $\pi(x_k)\, f = f(x_k)\, \omega_k$

$$\pi(x_k)\, h = \pi(x_k)\, f - \sum_{i=1}^{n} f(x_i)\, \pi(x_k)\, \omega_i$$
$$= f(x_k)\, \omega_k - \sum_{i=1}^{n} f(x_i)\, \omega_i(x_k)\, \omega_i$$
$$= 0$$

But $\omega_k(x_k) = 1$. Therefore, $\omega_k \notin Sp_\pi h \;\; \forall\, k,\ 1 \le k \le n$

In view of (1) we conclude that, $Sp_\pi h = \emptyset$.

Hence the corollary.

The following proposition gives a characterization for a multiplier of A in terms of spectral subspaces associated with regular representation.

3.4.8 Proposition:

A bounded linear operator T on A is a multiplier if and only if $T(M^{\lambda}(E)) \subseteq M^{\lambda}(E)$ for every closed set E in $\Delta(A)$.

Proof:

The 'only if' part can be easily seen by applying definitions.

We prove the 'if' part. Let T satisfy the hypothesis. Then the adjoint T^* of T also satisfies

$$T^*(M^{\overline{\lambda}}(E)) \subseteq M^{\overline{\lambda}}(E) \;\forall \text{ closed set } E \text{ in } \Delta(A).$$

In particular $T^*(M^{\overline{\lambda}}\{\omega\}) \subseteq M^{\overline{\lambda}}(\{\omega\}) \;\forall\; \omega$ in $\Delta(A)$.

By proposition 3.4.6, $T^*\{\omega\} = \alpha\omega$, for some $\alpha \varepsilon \mathbf{C}$.

Thus there exists a function $\varphi : \Delta(A) \to \mathbf{C}$ such that

$$T^*(\omega) = \varphi(\omega)\omega \quad \forall\; \omega \varepsilon \Delta(A).$$

We claim that $(T(x))^{\wedge} = \varphi . \hat{x}$ for every $x \varepsilon A$.

Let $x \varepsilon A$ then we have $\overline{\lambda(x)}\,(\omega) = \hat{x}(\omega)\,\omega$ and

$$\overline{\lambda}(T(x))(\omega) = (x, T^*(\omega))\,\omega$$

Therefore, $(T(x))^{\wedge}(\omega)\cdot\omega = \overline{\lambda}(T(x))(\omega).$

$$= \varphi(\omega)\;\hat{x}(\omega).\omega$$

Thus, $(T(x))^{\wedge} = \varphi\cdot\hat{x}$

Hence T is a multiplier with $\hat{T} = \varphi$. (Cf. Larsen:[30, Corollary 1.2.1]).

3.4.9 Remarks:

(i) Let us recall the Eymard algebra $A_2(G)$, discussed in 1.2.3, is a regular N algebra and its maximal ideal space is identified with G. The dual of $A_2(G)$ is the Von-Neumann algebra $PM_2(G)$ of G.

Observe that for any $g \in A_2(G)$ we have $Sp_\lambda g = \operatorname{supp} \hat{g} = \operatorname{supp} g$. Let T be an element of $PM_2(G)$. Then the support of T defined by Eymard [15] is precisely what we call the spectrum of T, with respect to the contragradient representation of regular representation of $A_2(G)$. Proposition 3.4.4 is proved by Eymard [15, Proposition 4.4] in this particular context. Refer Herz [27, Section 8] for the same kind of discussion as above for $A_p(G)$.

(ii) Let G be a locally compact abelian group. Let $(\lambda_p, L^p(G))$ be the usual left regular representation of $L^1(G)$ on $L^p(G)$ $1 \leq p < \infty$. Then the following are true,

(a) $Sp\,\lambda_p = \Gamma$ where Γ denotes the dual group of G.

(b) If $g \in L^1 \cap L^p(G)$, then $Sp_{\lambda_p}(g) = \operatorname{supp} \hat{g}$.

(a) follows because λ_p is faithful.

It can be easily seen by using propositions 3.4.1 and 3.4.2.

Section 3.5:

In this section, we decompose a B-space representation (π, X) of A in terms of the minimal spectral subspaces if the spectrum of π is discrete and discuss some of its consequences. Finally, we end this section by discussing the relation between the spectrum of (π, X) and the spectrum of its non-standard hull.

3.5.1 Proposition:

Let (π, X) be a B-space representation of **A**. Let ω be an isolated point in $Sp\,\pi$. Then there exists a projection P on X with range $M^\pi\{\omega\}$ such that P commutes with $\pi(x)$ for every $x \in A$. Actually P belongs to the algebra generated by A and the identity operator I on X.

Before getting into the proof let us observe the following remarks.

(1) Suppose that A has a unit. Then A/Ker π is a commutative B-algebra with unit whose maximal ideal space is h(Ker π) which in turn is Sp π by definition. Observe that Sp π is compact. By applying Shilov's Idempotent theorem (Refer Bonsal and Duncan [4, Chapter 21, Theorem 5]), we can choose an element $\tilde{b}$ in A/I such that $\hat{\tilde{b}} = \chi_{\{\omega\}}$ and $\tilde{b}$ is idempotent. (Here I denotes the ideal Ker π). Thus, there exists a $b \varepsilon A$ such that $\hat{b}/\text{Sp}\ \pi = \chi_{\{\omega\}}$ and, $\pi(b)$ is a projection on X.

(2) The representation (π, X) of A induces a B-space representation (π^o, X) of A_e (Refer 1.2.1 for the definition of A_e) defined as the following:

$$\pi^o((x,\alpha)) = \pi(x) + \alpha I, \quad \alpha \varepsilon \mathbb{C} \quad x \varepsilon A.$$

We remark that Sp π = Sp $\pi^o \cap \Delta(A)$. (Refer Domar and Lindahl[13]).

Now we prove the proposition.

Let A_e, (π^o, X) be as in Remark (2). Then ω is isolated in Sp π^o also. By Remark (1) there exists a $b \varepsilon A_e$ such that, $\pi^o(b)$ is a projection and $b/\text{Sp}\ \pi^o = \chi_{\{\omega\}}$.

If we denote $\pi^o(b)$ by P then the following claim will complete the proof.

<u>Claim</u>: $M^{\pi}(\{\omega\}) = P(X)$.

Let us first prove that, $M^{\pi}\{\omega\} = M^{\pi^o}\{\omega\}$.
By the definition, $M^{\pi^o}\{\omega\} \subseteq M^{\pi}\{\omega\}$.

Now on the other hand if ξ belongs to $M^{\pi}(\{\omega\})$ then by theorem 3.3.1. We have, $\pi(x)\xi = \omega(x)\xi \quad \forall\, x \varepsilon A.$ Therefore $\pi^{o}((x,\alpha))\xi = \pi(x)\xi + \alpha\xi = \omega(x)\xi + \alpha\xi = \omega((x,\alpha))\xi$. Therefore, ξ is an eigenvector for π^{o} associated with eigenvalue . Therefore, $\xi \varepsilon M^{\pi^{o}}\{\omega\}$.

We prove the claim. Let $\xi \varepsilon X$. Then

$$Sp_{\pi^{o}}(P(\xi)) = Sp_{\pi^{o}}(\pi^{o}(b)\xi) \subseteq (\text{supp}\,\hat{b}) \cap Sp_{\pi^{o}}(\xi)$$

by Remark 3.2.4.

$$\subseteq (\text{supp}\,\hat{b}) \cap Sp\,(\pi^{o})$$

$$\subseteq \{\omega\}.$$

Therefore, $P(\xi) \ \varepsilon \ M^{\pi^{o}}(\{\omega\}) = M^{\pi}(\{\omega\})$.

Conversely if $\xi \varepsilon M^{\pi}(\{\omega\})$, then $\pi^{o}(b)\xi = \omega(b)\xi$ i.e. $P(\xi) = \xi$. Therefore $M^{\pi}(\{\omega\}) \subseteq P(X)$.

Hence the claim.

Remarks:

(1) This proposition for the group algebra $L^{1}(G)$ is stated in Wolff [42].

(2) This proof is essentially due to Domar and Lindahl which is used in proving theorem 6.5.9 of [13].

3.5.2 Theorem:

Let (π,X) be a B-space representation of A. Suppose that the spectrum of π is discrete. Then the following assertions are true.

(i) For every ω in the spectrum of π, there exists a projection P_{ω} in the closed algebra generated by $\pi(A)$ and the identity in $L(X)$,.

(a) P_ω commutes with π and the range of P_ω is $M(\{\omega\})$.

(b) If we set, $\pi_\omega(x) = P_\omega \circ \pi(x) \circ P_\omega \ \forall\, x \varepsilon A$, then π_ω defines a sub-representation of π on $M(\{\omega\})$.

(ii) π is the direct sum of the sub-representations π_ω where ω ranges over Sp π. i.e. $\pi = \sum^{\oplus}_{\omega \in Sp\pi} \pi_\omega$ and $X = \sum^{\oplus}_{\omega \in Sp\pi} M^{\pi}(\{\omega\})$.

Proof:

(i) follows from the above proposition.

Now we prove (ii).

Claim: $X = (\sum^{\oplus}_{\omega \varepsilon Sp\pi} M(\{\omega\}))^{-}$

For every $\omega \ \varepsilon$ Sp π there exists V_ω a neighbourhood of ω such that $V_\omega \cap \mathrm{Sp}\,\pi = (\{\omega\})$. By (*) of corollary 3.3.6,

$0 \neq R(V_\omega) \subseteq M(\{\omega\})$.

Now it is easy to see that if ω, τ are in Sp π such that $\omega \neq \tau$ then $M(\{\omega\}) \cap M(\{\tau\}) = 0$.

Let $V = \bigcup_{\omega \varepsilon Sp\pi} V_\omega$, then V is a open neighbourhood of Sp π. We have,

$$\begin{aligned}(\sum^{\oplus}_{\omega \varepsilon Sp\pi} M(\{\omega\}))^{-} \subseteq X &= R(V) \quad \text{by } 3.2.2. \\ &= R(\bigcup_{\omega \varepsilon Sp\pi} V_\omega) \\ &= (\sum_{\omega \varepsilon Sp\pi} R(V_\omega))^{-} \\ &= (\sum^{\oplus}_{\omega \varepsilon Sp\pi} M(\{\omega\}))^{-}\end{aligned}$$

This claim proves (ii).

Hence the theorem.

3.5.3 Corollary:

Let (π,X) be a representation of A such that Sp π is discrete. An operator $T \in L(X)$ commutes with π iff,

$T(M(E)) \subseteq M(E)$ for every closed subset E of $\Delta(A)$.

If T commutes with π then it trivially satisfies the condition.

On the other hand let T satisfy the condition. Then, in particular, $T(M(\{\omega\}) \subseteq M(\{\omega\})$ $\forall$ $\omega \in$ Sp π, and so,

$$\pi(x)\circ T = T\circ\pi(x) \quad \text{on } M(\{\omega\})$$

for, $M(\{\omega\})$ contains only eigenvectors associated with ω.

Using the above theorem we conclude that T commutes with π.

3.5.4 Corollary:

Suppose that $\Delta(A)$ is discrete. If π_1 and π_2 are any two representations defined on X. Then,

$$\pi_1 \cong \pi_2 \quad \text{iff}$$
$$M^{\pi_1}(E) = M^{\pi_2}(E) \quad \forall \text{ closed subset E of } \Delta(A).$$

Proof:

It is clear by the above corollary.

Remark:

Arveson [1] proved the above corollary for $L^1(G)$: G being any locally compact abelian group not necessarily compact. It is discussed in Combes and Delaroche [6].

3.5.5 Corollary:

Assume that $\Delta(A)$ is discrete. Then A can be written as topological direct sum of minimal ideals. In other words

Socle of A exists and it equals A.
(For the definition of Socle refer Rickart [36], Chapter II; 1).

By applying the above theorem for λ, the regular representation, we observe that $\forall\ \omega$ there exists $a_\omega \in A$ such that a_ω is idempotent and $ba_\omega = \omega(b)a_\omega$ and using 3.4.3, we see that $M^\lambda(\{\omega\}) = Aa_\omega$, the minimal ideal containing a_ω.

In the remainder of the section we discuss the relation between spectrum of (π, X) and the spectrum of its non-standard hull.

<u>3.5.8</u> Theorem:

Let (π, X) be a B-space representation of A and $(\tilde{\pi}, \tilde{X})$ be non-degenerate non-standard hull of (π, X) defined in 2.3.1. Then $\mathrm{Sp}\ \pi = \mathrm{Sp}\ \tilde{\pi} = P_{\tilde{\pi}}$.
(Refer 3.2.6 for the definition of $P_{\tilde{\pi}}$).

<u>Proof</u>:

We claim that $\mathrm{Sp}\ \pi \subseteq P_{\tilde{\pi}}$.
By theorem 3.2.9, $\mathrm{Sp}\ \pi = AP_\pi$.
Let $\omega \in \mathrm{Sp}\ \pi$, then there exists a net $\{\xi_i\}_{i\in I}$ in X such that

$\|\xi_i\| = 1 \quad \forall\ i\in I$ and

$$\lim_i \|\pi(a)\xi_i - \omega(a)\xi_i\| = 0 \quad \forall\ a\in A \tag{1}$$

We proceed as we did in proposition 2.2.2.

By transfer principle there exists a net $\{\xi_i\}_{i\in {}^*I}$ in ${}^*X$ with $\|\xi_i\| = 1 \ \forall\ i\in {}^*I$, and thus $\xi_i \in \hat{X} \ \forall\ i$.

By concurrence theorem there exists a k such that $k \geq i \ \ \forall\, i\varepsilon I$. Let $a\varepsilon A$ and $\varepsilon>0$. There exists i_o such that,

$$\models \ (\forall\ i\varepsilon I)((i\geq i_o) \Rightarrow \|\pi(a)\xi_i - \omega(a)\xi_i\| < \varepsilon) \ \text{ by (1).}$$

By transfer principle,

$$* \models \ (\forall i\varepsilon {}^*I)((i\geq i_o) \Rightarrow \|{}^*\pi(a)\xi_i - \omega(a)\ \xi_i\| < \varepsilon).$$

Since $\xi_i \varepsilon \hat{X}$, ${}^*\pi(a)\ \xi_i = \hat{\pi}(a)\ \xi_i$ and $k \geq i \ \ \forall\, i\varepsilon I$, implies that

$$\|\hat{\pi}(a)\xi_k - \omega(a)\xi_k\| < \varepsilon \quad \forall\ \varepsilon>0 \text{ and } \forall\ a\varepsilon A.$$

Thus, $\|\hat{\pi}(a)\xi_k - \omega(a)\xi_k\| \approx 0$. Therefore,

$$\|\hat{\pi}(a)\xi_k - \omega(a)\xi_k\| = 0, \quad \forall\ a\varepsilon A.$$

Since $\hat{\pi}(a)\ \xi_k = \omega(a)\xi_k$, $\xi_k \varepsilon \tilde{X}$ and therefore,

$$\tilde{\pi}(a)\xi_k = \omega(a)\ \xi_k. \quad \text{Hence } \omega\ \varepsilon\ P_{\tilde{\pi}}.$$

On the other hand, if $\omega \notin Sp\ \pi$ then there exists an element x in A such that $\pi(x) = 0$ but $\omega\ (x) \neq 0$. Since $\pi(x) = 0$, we have $\tilde{\pi}(x) = 0$. Therefore $\omega \notin Sp\ \tilde{\pi}$. This proves the $Sp\ \tilde{\pi} \subseteq Sp\ \pi$.

Thus, $Sp\ \tilde{\pi} \subseteq Sp\ \pi \subseteq P_{\tilde{\pi}}$. But we know that $P_{\tilde{\pi}} \subseteq Sp\ \tilde{\pi}$.

Hence the theorem.

<u>Remark</u>:

For the group algebra the above theorem was proved by Wolff [42]. Finally, we remark that in Wolff [41] and Greiner and Groh [22] a lot of problems in connection with spectral subspaces of Banach lattice representations of locally compact abelian groups are considered.

CHAPTER 4

APPLICATIONS

The main purpose of this chapter is to give various applications of the theory we developed in Chapter 3.

If (π, X) is a B-space representation of a commutative regular N algebra (see 1.2.3 for the definition), then in Section 4.1 we shall give sufficient conditions for $L_{\pi}(X)$ to be identified with $L_{\bar{\pi}}(X^*)$ in terms of spectrum of π and spectral subspaces associated with π. As a corollary we prove that if the maximal ideal space $\Delta(A)$ is discrete, then $L_{\bar{\lambda}}(A^*)$ is identified with the algebra of multipliers of A. We also prove the converse of the above corollary when A is further assumed to be M-regular for some $M > 0$. We discuss the above results for concrete examples. In Section 4.2 we prove that any norm closed A-invariant subspace of A^*, where A is a commutative M-regular N algebra, is reflexive if and only if it is finite dimensional.

Section 4.1:

We need the following fact in elementary Banach space theory. Since we could not find a proof in the literature, we have included a proof here.

4.1.1 Lemma:

Let X be a B-space. Let T be a bounded linear operater on X^*. Then T is weak $*$ -weak $*$ continuous if and only if T

satisfies the following.

If for every net $\{f_i\}_{i\varepsilon I}$ in X^* such that $||f_i|| \leq r$ for some $r > 0$ and $\{f_i\}$ converges to 0 in weak-* topology then $\{T(f_i)\}$ converges to 0 in weak-* topology.

Proof:

We define a locally convex topology on X^*, coarser than the weak-* topology, as follows.

Let U be a set in X^* containing 0. We say that U is an open neighbourhood around zero iff for every $M > 0$ $U \cap B_M(0)$ is open in the weak-* topology, where $B_M(0)$ denotes the set $\{f\varepsilon X: ||f|| \leq M\}$.

The following result is found in Dunford and Schwartz [14, V. 5.6].

If θ is a linear functional on X^*, then θ is continuous in the above topology iff θ is continuous in the weak-* topology.

Now we prove the lemma. Let $T \varepsilon L(X^*)$ satisfy the hypothesis. To prove T is weak* - weak * continuous, we need to prove that for every $x\varepsilon X$, $\theta_x: f \to (x,T(f))$ is a weak-* continuous linear functional, i.e. we need to prove that θ_x is continuous in the above topology which is of course, true by the hypothesis.

This proves the lemma.

Let A be a B-algebra. (π,X) be a B-space representation. Let $T \varepsilon L_\pi(X)$ i.e., $T \varepsilon L(X)$ and, $T\circ\pi(a) = \pi(a)\circ T$, $\forall$ $a\varepsilon A$. Then, $T^*: A^* \to A^*$ is easily seen to satisfy

$$T^* \circ \bar{\pi}(a) = \bar{\pi}(a) \circ T^*, \forall a\varepsilon A.$$

Thus, there exists an isometric isomorphism from $L_\pi(X)$ into $L_{\bar{\pi}}(X^*)$ through the usual transpose map. The following theorem gives sufficient conditions for the above map to be surjective.

4.1.2 Theorem:

Let A be a commutative regular N algebra and let (π, X) be a non-degenerate B-space representation of A. Assume that

(i) Sp π is discrete.

(ii) $\forall\ \omega \in$ Sp π, the dimension of $M^\pi(\{\omega\})$ is at most 1.

Then $L_{\bar{\pi}}(X^*) \cong L_\pi(X)$. In fact, we prove that any operator on X^* commuting with the action of A can be realized as the transpose of an operator on X commuting with the action of A. (For the definition of Sp π, $M^\pi(E)$, refer Section 3.1).

Proof:

Since Sp π is discrete, we invoke theorem 3.5.2 to decompose π as $\displaystyle\sum_{\omega \in \mathrm{Sp}\,\pi}^{\oplus} \pi_\omega$. We have,

$$X = \sum_{\omega \in \mathrm{Sp}\,\pi}^{\oplus} M^\pi(\{\omega\}) \quad \text{and} \quad X^* = \sum_{\omega \in \mathrm{Sp}\,\pi}^{\oplus} M^{\bar{\pi}}(\{\omega\}).$$

Moreover, for every ω, we have a projection P_ω on X such that $0 \neq M^\pi(\{\omega\}) = P_\omega\,(X)$, and $P_\omega \circ \pi(a) = \pi(a) \circ P_\omega$. By (ii) $M^\pi(\{\omega\})$ is 1-dimensional subspace $\forall\ \omega \in$ Sp π. Fix a $\xi_\omega \neq 0$ in $M^\pi(\{\omega\})$ such that $M^\pi(\{\omega\})$ is spanned by ξ_ω.

Chaim 1:

$M^{\bar{\pi}}(\{\omega\}) = \{\eta \in X^*: (\xi_\omega, \eta) \neq 0\}$ and therefore $M^{\bar{\pi}}(\{\omega\})$ is also 1-dimensional.

For, if $\eta \in M^{\bar{\pi}}(\{\omega\})$ and $\eta \neq 0$ then $Sp_{\bar{\pi}}(\eta) = \{\omega\}$. If V is a open neighbourhood of ω such that $V \cap Sp\,\pi = \{\omega\}$, then by theorem 3.4.1 there exists a $\xi \neq 0$ such that $Sp\,\xi \subseteq V$ and $(\xi, \eta) \neq 0$.

But $Sp\xi \subseteq Sp\,\pi$. So, $Sp\,\xi \subseteq \{\omega\}$, but $\xi \neq 0$. Therefore, $Sp\,\xi = \{\omega\}$. By the above, $\xi = \alpha\xi_\omega$ for some $\alpha \varepsilon \mathbb{C}$. Thus $(\xi,\eta) \neq 0$. Similarly we can prove the other way.

Fix a $\eta_\omega \in M^{\bar{\pi}}(\{\omega\})$ such that $(\xi_\omega, \eta_\omega) = 1$. Then $M^{\bar{\pi}}(\{\omega\})$ is the 1-dimensional space spanned by η_ω. For, if $\eta \in M^{\bar{\pi}}(\{\omega\})$ then observe that $Sp\,(\eta - (\xi_\omega, \eta)\eta_\omega) = \emptyset$ so that $\eta = (\xi_\omega, \eta)\eta_\omega$. Thus the claim 1.

If $T \in L_{\bar{\pi}}(X^*)$, then $T(M^{\bar{\pi}}(\{\omega\})) \subseteq M^{\bar{\pi}}(\{\omega\})$ $\forall\, \omega \in Sp\,\pi$, by the definition of the spectral subspaces.

Therefore, there exists a function $\hat{T}$ on $Sp\,\pi$ such that,

$$T(\eta_\omega) = \hat{T}(\omega)\,\eta_\omega$$

For each $\omega \in X^*$ we define $\hat{\eta}$ on $Sp\,\pi$ by $\hat{\eta}(\omega) = (\xi_\omega, \eta)$.

<u>Claim 2</u>:

For every $T \in L_{\bar{\pi}}(X^*)$, $T \circ P^*_\omega = P^*_\omega \circ T$ $\forall\, \omega \in Sp\,\pi$. For, let $C_T = \{S \in L(X^*) : S \circ T = T \circ S\}$. Then, C_T is closed with respect to the strong operator topology. Identity operator is easily seen to be in C_T. Since $T \in L_{\bar{\pi}}(X^*)$, $\bar{\pi}(a) \in C_T$ $\forall\, a \varepsilon A$. Using theorem 3.5.2 we observe that P^*_ω is in the norm closure of the algebra generated by $\bar{\pi}(A)$ and I.

Hence $P^*_\omega \in C_T$. This proves the claim 2.

Claim 3:

For every $\eta \in X^*$ and $T \in L_{\overline{\pi}}(X^*)$, we have the following.

(i) $P_\omega^* (\eta) = \hat{\eta}(\omega)\, \eta_\omega$

(ii) $(T(\eta))^\wedge (\omega) = \hat{\eta}(\omega)\, \hat{T}(\omega)$.

Let $\xi \in X$. Then $\xi = P_\omega(\xi) + (1-P_\omega)\xi$.

There exists a complex number α such that $P_\omega \xi = \alpha \xi_\omega$

$$(\xi, \eta_\omega) = (\xi, P_\omega^*(\eta_\omega)) = \alpha(\xi_\omega, \eta_\omega) = \alpha$$

Therefore,

$$(\xi, P_\omega^*(\eta)) = (P_\omega(\xi), \eta) = (\xi, \eta_\omega)(\xi_\omega, \eta) = (\xi, \eta_\omega)\,(\hat{\eta})(\omega)$$

Hence $P_\omega^*(\eta) = \hat{\eta}(\omega)\eta_\omega$

(ii) $$\begin{aligned}(T(\eta))^\wedge(\omega) &= (\xi_\omega, T(\eta)) = (P_\omega(\xi_\omega), T(\eta)) \\ &= (\xi_\omega, P_\omega^* \circ T(\eta)) = (\xi_\omega, T \circ P_\omega^*(\eta)) \\ &= \hat{\eta}(\omega)(\xi_\omega, T(\eta_\omega)) = \hat{\eta}(\omega)(\xi_\omega, \hat{T}(\omega)\eta_\omega) \\ &= \hat{T}(\omega)\, \hat{\eta}(\omega).\end{aligned}$$

Hence the claim 3.

We have thus shown that for every $T \in L_{\overline{\pi}}(X^*)$ and $\eta \in X^*$ there exist mappings $\hat{T}$ and $\hat{\eta}$ on $\mathrm{Sp}\,\pi$ such that,

$$(T(\eta))^\wedge = \hat{T}\,\hat{\eta} \qquad \text{on } \mathrm{Sp}\,\pi.$$

To complete the proof of the theorem, it is sufficient to show that the transpose map from $L_\pi(X)$ into $L_{\overline{\pi}}(X^*)$ is surjective, i.e. we need to prove that any $T \in L_{\overline{\pi}}(X^*)$ is weak*- weak* continuous. We use the lemma 4.1.1 to accomplish this.

Let $T \in L_{\overline{\pi}}(X^*)$. Let $\{\eta_i\} \subseteq X^*$ be a net such that $\|\eta_i\| \leq M \ \forall\ i$ and $\{\eta_i\}$ converges to 0 in the weak-* topology.

Claim 4:

The net $\{T(\eta_i)\}$ converges to 0 in the weak-* -topology. Observe that every $\omega \ \varepsilon \ Sp\ \pi$, $(\xi_\omega, \eta_i) \to 0$ i.e. $\hat{\eta}_i(\omega) \to 0$. and therefore, $(\xi_\omega, T(\eta_i)) = \hat{T}(\omega)\, \hat{\eta}_i(\omega) \to 0$.

Thus, if ξ is of the form $\sum_{i=1}^{n} \alpha_i \ \xi\omega_i$ where

where $\omega_i \ \varepsilon\ Sp\ \pi$, $\alpha_i \ \varepsilon\ \mathbb{C}$; then $(\xi\ ,\ T(\eta_i)) \to 0$. Since $X = \sum^{\oplus}_{\omega\varepsilon\ Sp\pi} M^\pi(\{\omega\})$, we have, for any $\xi \ \varepsilon\ X$ there exists a net $\{\zeta_j\}$ which are finite linear sums as above such that $\zeta_j \to \xi$ in norm. Then,

$$|(\zeta,\ T(\eta_i))| \ \leq\ |(\zeta_j,\ T(\eta_i))| + \|\zeta_j - \xi\|\ \|T(\eta_i)\|$$

$$\leq\ |(\zeta_j,\ T(\eta_i))| + \|\zeta_j - \xi\|\ \|T\|\ M.$$

Therefore, by taking the limits, we see that

$$\lim_i |(\xi,\ T(\eta_i))| = 0.$$

This proves the claim 4.

Hence the theorem.

4.1.3 Corollary:

Let A be a commutative regular N algebra. Suppose that the maximum ideal space $\Delta(A)$ is discrete. Then, $L_{\overline{\lambda}}(A^*)$ is identified with the multipliers of A, by the usual transpose map.

Using 3.4.3 (ii), we see that the regular representation satisfies the hypothesis of the theorem.

Remark 1:

Recall that, for regular representation λ, and for every $\omega \in \Delta(A)$, $M^{\overline{\lambda}}(\{\omega\})$ is 1-dimensional subspace spanned by ω (Cf. Corollary 3.4.5). Therefore for any A, for which $\Delta(A)$ is not necessarily discrete, and for any $T \in L_{\overline{\lambda}}(A^*)$ we can always define $\hat{T}: \Delta(A) \to \mathbb{C}$ by $T(\omega) = \hat{T}(\omega)\,\omega$ for, $T(M^{\overline{\lambda}}(\{\omega\})) \subseteq M^{\overline{\lambda}}(\{\omega\})$ as in the proof of the above theorem.

Let us recall a well known result in the multiplier theory of semisimple commutative Banach algebras. The multiplier algebra M(A) is linearly isomorphic with space of all continuous bounded functions φ on $\Delta(A)$ such that $\varphi \cdot \hat{A} \subseteq \hat{A}$. The function associated with any $T \in M(A)$ is usually denoted by $\hat{T}$ and it satisfies $T^*(\{\omega\}) = \hat{T}(\omega)\,\omega$ (Refer for example Larsen [30]). This prompted us to use the same notation for any T in $L_{\overline{\pi}}(X^*)$ in the above theorem and in particular any T in $L_{\overline{\lambda}}(A^*)$.

Remark 2:

It is well known that the above Corollary 4.1.3 and its converse are true for $L^1(G)$: G a lca group. Taking the clue from this algebra, we prove that the converse is also true under some additional hypothesis on A which of course, the group algebra enjoys.

4.1.4 Theorem:

Let A be a commutative M-regular N algebra for some $M>0$. Suppose that , the usual transpose map taking the multiplier algebra M(A) into $L_{\bar{\lambda}}(A^*)$ is surjective. Then the maximal ideal space $\Delta(A)$ is discrete.

For proving the theorem, we need the following lemma.

4.1.5. Lemma:

Let $\omega \varepsilon \Delta(A)$ be fixed. Then there exists a net $\{a_i\}$ in A satisfying the following:

(i) $\|a_i\| \leq M$ and $\omega(a_i) = 1 \quad \forall\, i$

(ii) For every $a\varepsilon A$, $\lim\limits_i \|aa_i - \omega(a)\ a_i\| = 0$.

Notice that $AP_\lambda = Sp\,\lambda = \Delta(A)$ where AP_λ denotes the approximate point spectrum of λ. (Refer 3.2.7 and the theorem 3.2.9) Therefore, for any $\omega \varepsilon \Delta(A)$ we do get a net satisfying (ii), but we need to show that $\omega(a_i) = 1$, for which we provide a separate proof.

Proof:

Let $\{U_i\}$ denote the neighbourhood system at ω. Using the M-regurarity, choose a_i in A satisfying (i) of the lemma and such that support of $\hat{a}_i \subseteq U_i$. Now we verify that this net $\{a_i\}$ satisfies the condition(ii).

Let $a\varepsilon A$ and $\varepsilon>0$.

Case(i): $\omega(a) = 0$.

Since $\{\omega\}$ is a S-set there exists a compact neighbourhood U of ω and $b\varepsilon A$ such that $\hat{b}/U = 0$ and $\|a-b\| < \varepsilon/M$.

Thus $a_i b = 0$ for large i and so (ii) obtains.

Case (ii): $\omega(a) \neq 0$.

Let the supp $\hat{a}$ be compact. Denote this by K.
Let $b \in A$ be such that $\hat{b}/K = \omega(a)$. If $x = a-b$ then $\omega(x) = 0$ and so there exists a neighbourhood U of ω and $y \in A$ such that $\|x-y\| < \varepsilon/M$ and $\hat{y}/U = 0$.

If i is so large that $U_i \subseteq U \cap K$, then we have $ya_i = 0$ and $ba_i = \omega(a)a_i$. Therefore,
$\|aa_i - \omega(a)a_i\| = \|aa_i - ba_i - ya_i\| = \|(a-b)a_i - ya_i\| < \varepsilon$

Hence the lemma.

Since A is Tauberian, (ii) is true for any $a \in A$.

4.1.6 Lemma:

Let A be a commutative M-regular N algebra for some $M>0$. Let ω be in $\Delta(A)$. Then there exists an element F in A^{**} such that

(i) $(\omega, F) = 1$.

(ii) For every $f \in A^*$, $a \in A$, $(\overline{\lambda}(a)f, F) = \omega(a)(f, F)$.

Proof:

Choose a net $\{a_i\}$ satisfying the conditions given in the previous lemma. By Banach-Alaoglu's theorem, there exists a subnet of $\{a_i\}$, which we again denote by $\{a_i\}$ and an element F in A^{**} such that $\{a_i\}$ converges to F in A^{**} with respect to the weak-$*$ topology. (Here a_i is considered to be an element in A^{**}).

Therefore, $\forall\ f \in A^*$, $\lim\limits_{i} (f, a_i) = (f, F)$.

In particular, $(\omega, F) = \lim\limits_{i} (\omega, a_i) = 1$.

For every $f \varepsilon A^*$, $a \varepsilon A$, we have

$$(\bar{\lambda}(a)f, F) = \lim_i (\bar{\lambda}(a)f, a_i) = \lim_i (a_i, \bar{\lambda}(a)f)$$

$$= \lim_i (aa_i, f) = \lim_i (\omega(a)a_i, f)$$

by (ii) of the above lemma.

$$= \omega(a)(f, F), \text{ as we required.}$$

Hence the lemma.

Now we prove the theorem:

Suppose $L_{\bar{\pi}}(A^*) \cong M(A)$. To conclude that $\Delta(A)$ is discrete, we prove that $\chi_{\{\omega\}}$ is a continuous function on $\Delta(A)$ for every ω, in $\Delta(A)$. Actually we are going to show that there exists a $T \varepsilon M(A)$ such that $\hat{T} = \chi_{\{\omega\}}$.

By hypothesis, it is enough to show that there exists a $T \varepsilon L_{\bar{\lambda}}(A^*)$ such that $\hat{T} = \chi_{\{\omega\}}$, where $\hat{T}$ is defined in Remark (i) after the corollary 4.1.3.

Let $F \varepsilon A^{**}$ be as in Lemma 4.1.6.
Define $T: A^* \to A^*$ by $T(f) = (f, F)\omega$. Then T is easily seen to be a bounded linear operator on A^*. T belongs to $L_{\bar{\lambda}}(A^*)$ also. For,

if $a \varepsilon A$, $f \varepsilon A^*$, then,

$$T(\bar{\lambda}(a)f) = (\bar{\lambda}(a)f, F)\omega = \omega(a)(f, F)\omega .$$

and, $\bar{\lambda}(a)(T(f)) = \bar{\lambda}(a)((f, F)\omega) = (f, F)\bar{\lambda}(a)(\omega)$

$$= (f, F)\omega(a)\omega .$$

Let us now calculate $\hat{T}$.

It is defined by $T(\tau) = \hat{T}(\tau)\,\tau \quad \forall\ \tau \in \Delta(A)$.

Let $a \in A$, $\tau \in \Delta(A)$. Then

$$\hat{T}(\tau)\ \tau(a) = (a, T(\tau)) = F(\tau)\,\omega\,(a)$$

If $\tau \neq \omega$, choose an element $a \in A$ such that $\tau(a) = 1$ and $\omega(a) = 0$

then, $\hat{T}\ (\tau) = 0$.

If $\tau = \omega$ then, $\hat{T}(\omega)\,\omega(a) = F(\omega)\,\omega(a) = \omega(a)$.

Therefore, $\hat{T}(\omega) = 1$.

Thus, $\hat{T} = \chi_{\{\omega\}}$ as we promised to prove.

This completes the proof of the theorem.

The following is an interesting corollary of Lemma 4.1.6.

4.1.7 Corollary:

Let A be as in the lemma 4.1.6. Then for every ω in $\Delta(A)$, there exists F_ω in A^{**} such that $F_\omega / \Delta(A) = \chi_{\{\omega\}}$.

Proof:

Let us denote the F we got in the lemma 4.1.6 by F_ω. Then by (ii) of the lemma, we have,

$\forall\ \tau \in \Delta(A)$; $\forall\ a \in A$, we have,

$$(\bar{\lambda}(a)\ \tau, F_\omega) = \omega\ (a)(\tau, F_\omega)$$

i.e., $\tau\,(a)(\tau, F) = \omega\,(a)(\tau, F_\omega)$.

This proves the corollary.

Remark:

The following theorem summarizes what we have proved so far.

4.1.8 Theorem:

Let A be a commutative M-regular N algebra for some $M > 0$. Then the maximal ideal space is discrete if and only if the algebra of all operators in $L(A^*)$ commuting with $\bar{\lambda}(A)$ can be identified with the algebra of multipliers through the transpose map.

In Chapter 5, we shall give some more sufficient conditions for the algebras of above kind to have discrete maximal ideal space.

Section 4.2:

Let A be a B-algebra. We say a subspace X in A^* is invariant if for every $a \in A$, $\bar{\lambda}(a)X \subseteq X$, holds.

4.2.1 Theorem:

Let A be a commutative M-regular N algebra for some $M>0$. Suppose that X is a norm closed invariant subspace of A^*. Then X is reflexive if and only if X is finite dimensional.

For proving the theorem, we need the following Lemma.

4.2.2 Lemma:

Let J be a set contained in $\Delta(A)$. Then J is relatively weakly compact as a subset of A^* if and only if J is finite.

Proof:

Let $J \subseteq \Delta(A)$ and let K denote the weak closure of J in A^*. Assuming that K is weakly compact, we prove that J is a finite set. Observe that K is actually contained in $\Delta(A)$.

Suppose that J is not a finite set.

Then there exists a sequence $\{\tau_n\}$ in F which are distinct. Since K is weakly compact, by Eberlein-Smulian Theorem (Cf. Dunford and Schwartz [14, V.6.1]), there exists a subsequence of $\{\tau_n\}$ which we again denote by $\{\tau_n\}$, and a τ in K such that

$\{\tau_n\}$ converges to τ weakly.

Observe that K is weak-* compact also. Therefore it is a compact subset of $\Delta(A)$, in the Gelfand topology. By the regularity of A there exists an element $a \varepsilon A$ such that $\hat{a}/K = 1$. In particular, $\hat{a}(\tau) = 1$, since $\tau \varepsilon K$ and therefore $\tau \neq 0$.

We know that for every $\omega \varepsilon \Delta(A)$, there exists $F \ \varepsilon\ A^{**}$ such that $F/\Delta(A) = X_{\{\omega\}}$ by Corollary 4.1.7.

Since no τ_n is equal to τ, we have $(\tau_n, F) = 0$, $\forall$ n.
But $(\tau, F) = 1$. This contradicts the fact that τ_n converges to τ weakly.

Hence the Lemma.

We now prove the theorem.

Let X be a reflexive invariant subspace of A^*, which is norm closed. Let S denote the unit ball of X. Then S is weakly compact, since X is reflexive. Therefore on S weak and weak-* topologies coincide. Thus S is weak-* closed in A. By Krein-Smulian theorem (Cf. Dunford and Schwartz [14, V.5.7]), X is a weak-* closed subspace.

Since X is invariant, denote the sub-representation of $\bar{\lambda}$ restricted to X by ρ.

By Corollary 3.4.7, $\Delta(A) \cap X$ is non-empty, Since $\Delta(A) \cap X$ is a weak-$*$ closed subset of S, it is weakly compact also, because on S, both the topologies coincide. Therefore by the above lemma $\Delta(A) \cap X$ is a finite set.

Again by the Corollary 3.4.7 (ii), X is finite dimensional.

Hence the theorem.

Remarks:

(1) The above result was proved by Glicksberg [18] for the group algebra of a locally compact abelian or a compact non-abelian group.

(2) E.E. Granirer [21] proved the above result for $A_p(G)$, $1<p<\infty$ where G is an amenable group. But recall that $A_p(G)$ where G is any locally compact group, not necessarily amenable, are 1-regular N algebras (cf. 1.2.3). Thus our result includes the contributions of Glicksberg and Granirer.

CHAPTER 5

INTROVERTED SUBSPACES AND THEIR DUALS

It is well known that A^{**}, the bidual of a Banach algebra A is again a Banach algebra with any of the Arens Products. Moreover, Arens product extends the multiplication defined on A. But A^{**} is too large to retain some of the basic properties A already enjoys. For example, even for such good algebras as $L^1(G)$, G an infinite locally compact abelian group, $(L^1(G))^{**}$ is neither commutative nor semisimple. And this motivates us to consider certain quotient algebras of A^{**} which contain A isomorphically. In otherwords, these are the algebras which extend the multiplication defined on A. These algebras are, actually, duals of certain A-invariant subspaces of A^*, which are known as introverted subspaces. Specific examples of introverted subspaces were studied by various authors (we will be discussing these examples in Sec. 5.1) and A.T. Lau [32] studied them for a particular B-algebra namely $A_2(G)$. The aim of this chapter is to present a unified approach to the study of introverted subspaces on an arbitrary B-algebra. We shall discuss some of the applications of these studies.

In Section 5.1, we fix-up some notations, define introverted subspaces, discuss its connection with quotient algebras of A^{**}, and give some examples. In Sec. 5.2 we study the

commutative Banach algebras. In this special case, we shall give some more examples. Starting with a commutative B-algebra A, it is quite interesting to know how far the commutativity of A can be extended to these quotient algebras. We give a definite solution to this problem generalizing the result obtained by A.T. Lau [32] for $A_2(G)$ in theorem 5.2.7.

In Sec. 5.3, we take up the case of a Banach algebra, not necessarily commutative, but having bounded approximate identity.If $B=X^*$ is a quotient algebra, for any introverted subspace X, then, we imbed the algebra of left multipliers $M_\ell(A)$ in B and give a criterion for any F in B to arise from $M_\ell(A)$, and discuss its connection with an earlier result of Birtel [3]. Lastly, we give a functional characterization for the algebra of A-commuting operators on X. The final section of this chapter is for applications. In Theorem 5.4.3 we prove that, for a commutative M-regular N algebra A with $\Delta(A)$ containing no isolated points, every weakly compact multiplier is zero. Theorem 5.4.4 gives several characterizations for A of above kind to have discrete maximal ideal space.

Section 5.1:

Let us recall that the Banach algebra A is always assumed to be right faithful, (Cf.1.2.1). Arbitrary elements of A; A^*; A^{**} will usually be denoted by a,b,c,x,y,...; f,g,...; and E,F,G,... respectively.

Let i denote the usual canonical imbedding from A into A^{**}. We often fail to write the map i explicitly i.e., A will rather be considered as a subspace of A^{**}.

5.1.1 Definition:

For $F, G \in A^{**}$, $f \in A^*$, $a, b \in A$, we define,

(i) fa in A^* by $(b, fa) = (ab, f)$

(ii) Gf in A^* by $(a, Gf) = (fa, G)$

(iii) $F \cdot G$ in A^{**} by $(f, F \cdot G) = (Gf, F)$.

Remarks:

(1) Note that $fa = \bar{\lambda}(a)f$, where $\bar{\lambda}$ is the contragradient representation of the left regular representation of A. We denote it this way for the sake of convenience. Similarly, we denote the linear function $b \to (ba, f)$ by af. If $G = a$ for some a in A, then $Gf = af$.

(2) Now (iii) defines a product in A^{**} which is known as Arens product. With this product A^{**} becomes a Banach algebra and this product extends the multiplication defined on A.

5.1.2 Definition:

(i) A subspace $X \subseteq A^*$ is called left invariant if for every $a \in A$, $Xa \subseteq X$.

(ii) A left invariant subspace X in A^* is called left introverted if for every F in A^{**}, and $f \in X$, Ff belongs to X.

Similarly we can define right invariant and right introverted subspaces. If A is commutative, then we simply call these subspaces as invariant and introverted subspaces.

5.1.3 Examples:

We give some examples of left introverted subspaces. Let A be a B-algebra.

(1) X = linear span of $\{fa: f\varepsilon A^*;\ a\varepsilon A\}$ then X is trivially left invariant. It is left introverted also for,

if $F \varepsilon A^{**}$, $g\varepsilon X$ is of the form fa for some $f\varepsilon A^*$, $a\varepsilon A$ then, $Fg = F(fa) = (Ff)a$ and $Ff \varepsilon A^*$. Therefore, $Fg \varepsilon X$. We denote this space by $\langle A^*A\rangle$. More generally, if X is a left introverted subspace of A^* then Y = linear span of $\{fa: f\varepsilon X,\ a\varepsilon A\}$ is again a left introverted subspace.

For $A = L^1(G)$; G, a lc group $\langle A^*A\rangle$ is none other than, $C_{ru}(G)$, the space of all bounded right uniformly continuous functions on G.

(2) Let X be the space of all functionals of the form,

$$\sum_{n=1}^{\infty} f_n x_n, \quad \text{where } f_n \varepsilon A^*;\ x_n \varepsilon A\ \forall\ n \text{ and } \sum_{n=1}^{\infty} \|f_n\|\,\|x_n\| < \infty.$$

Then X is a left introverted subspace.

(3) Let (π, Y) be a Banach space representation of A such that

$\|\pi(a)\| \leq \|a\|\ \forall\ a\varepsilon A$ (Refer 1.2.4). Let

$X = \{f\varepsilon A : |f(x)| \leq K\ \|\pi(x)\|$ for every $x\varepsilon A$, for some $K>0\}$.

Then X is a left introverted subspace. For,

if $a\varepsilon A$, $f\varepsilon X$, $F\varepsilon A^{**}$, and $K>0$ are such that

$|f(x) \leq K\ \|\pi(x)\|\ \forall\ x\varepsilon A$, then we have $\|fa\| \leq K\ \|\pi(a)\|$

$|(a, Ff)| \leq (K\|F\|)(\|\pi(a)\|)\ \forall\ a\varepsilon A$.

(4) Let A be a commutative B-algebra. Let $X = [\Delta(A)]$ i.e. the linear subspace spanned by its maximal ideal space $\Delta(A)$. Then since, $\omega a = \omega(a)\omega$ and $F\omega = (\omega, F)\,\omega$, we can easily see that X is introverted.

It is well known that if $A = L^1(G)$, G a lca group, then the norm closure of X is precisely the space of all almost periodic functions on G.

Remarks:

(i) The introverted subspaces were studied by A.T. Lau [32] for the algebra $A_2(G)$; G a lc group. The introverted subspace discussed in Example 1 has been extensively studied by Grosser and Losert [24]. Examples (2) and (4) were defined by Maté [34] and Birtel [3] respectively.

(ii) We donot loose any generality in assuming that a left introverted subspace is norm closed. Henceforth, any left introverted subspace is always assumed to be norm closed. In the next section, we give some more examples of introverted subspaces when, A is a commutative B-algebra.

5.1.4 Proposition:

Let X be a left introverted subspace of A^*. Then.

(i) X^* is a quotient algebra of A^{**}.

(ii) There exists a norm decreasing linear isomorphism from A into X^* iff X is total in A^*.

Proof:

We prove (i). Let us first show that $X^{\perp}$ is a closed two sided ideal in A^{**}.

Let $f \varepsilon X$; $F \varepsilon X^{\perp}$; $G \varepsilon A^{**}$. Then,

$(f, F \cdot G) = (Gf, F) = 0$ since $Gf \varepsilon X$.

Since X is left invariant $Ff = 0$ and so,

$(f, G \cdot F) = (Ff, G) = 0$.

Therefore, $F.G$, $G.F$ both are in $X^{\perp}$.

Now $X^* = A^{**}/X^{\perp}$ and so X^* is a quotient algebra of A^{**}. Denote this Banach algebra by B_X.

(ii) is easy to check.

5.1.5 Remarks:

(i) Examples (1) and (2) of 5.1.3 are total. Example (3) is total if π is a faithful representation and Example (4) is total if A is semisimple.

(ii) For our further investigations, we always assume that X is total in A^*. We shall henceforth denote B_X by B unless there arises a need to do so. Any general element of B will still be denoted by F,G,...

(iii) If A is commutative, and X is as in 5.1.3 (4) then, B_X is known as Birtel algebra.

(iv) If X and Y are two left introverted subspaces such that $X \subseteq Y$ then there exists a norm decreasing linear homomorphism from B_Y onto B_X.

Section 5.2:

Throughout this section, A is assumed to be a commutative semisimple B-algebra. In this section, we give a necessary and sufficient condition for a closed invariant subspace to be introverted and using this we give some more examples. Next, we give a characterization for B_X to be commutative in terms of X. Finally we prove a result about the inclusions of various introverted subspaces.

For any $f \epsilon A^*$, denote the convex set $\{fa: a \epsilon A, \|a\| \leq 1\}$ by $O(f)$.

5.2.1 Proposition:

Let X be a norm closed invariant subspace of A^*. Then X is introverted iff for every $f \epsilon X$, the weak-* closure of $O(f)$ is contained in X.

Proof:

Since X is invariant, $O(f) \subseteq X$ for every $f \epsilon X$. Now suppose that X is introverted.

Let $f \epsilon X$, $g \epsilon A$ such that g is in weak-* closure of $O(f)$, i.e., there exists a net $\{x_i\}$ in A such that $\|x_i\| \leq 1$ and $\{fx_i\}$ converges to g in the weak-* topology. Use Banach-Alaoglu's theorem, to obtain a subnet, which we again denote by $\{x_i\}$ and F in A^{**} such that $\{x_i\}$ converges to F in weak-* topology.

We prove that $g \epsilon X$ by showing that $Ff = g$.

$$\text{If } a \epsilon A, \ (a, Ff) = (fa, F) = \lim_i (fa, x_i)$$

$$= \lim_i (a, fx_i) = (a,g) \text{ as we required.}$$

We prove the converse. Let $F \epsilon A^{**}$, $f \epsilon X$.

We assume that $\| F \| \leq 1$. By Goldstine's theorem (Dunford and Schwartz [14, V.4.2], there exists a net $\{x_i\}$ in A such that $\|x_i\| \leq 1$ and $\{x_i\}$ converges to F in the weak-* topology.

Now $fx_i \in O(f)$ and,

$$\lim_i (a, fx_i) = \lim_i (x_i, fa) = (fa, F) = a, Ff).$$

Hence, $Ff \in$ weak-* closure of $O(f)$ and so $Ff \in X$ by hypothesis. Thus, X is introverted.

<u>5.2.2</u> Definition:

We say that a functional f in A^* is compact (weakly compact) if $O(f)$ is relatively compact (weakly compact) in A^*. The set of compact (weakly compact) functionals is denoted by CF (WCF). It can be easily seen that CF and WCF are norm closed invariant subspaces of A^*. We also have that the $[\Delta(A)] \subseteq CF \subseteq WCF$.

<u>5.2.3</u> Theorem:

Any norm closed invariant subspace X contained in WCF is introverted.

<u>Proof</u>:

LetX be a norm closed in variant subspace contained in WCF. Let $f \in X$. Since $O(f)$ is convex, norm closure of

$$\begin{aligned} O(f) &= \text{Weak closure of } O(f) \\ &= \text{Weak-* closure of } O(f) \end{aligned}$$

since $O(f)$ is relatively weakly compact. Therefore, weak-* closure of $O(f) \subseteq X$ and hence X is introverted by proposition 5.2.1.

5.2.4 Corollary:

CF, WCF are introverted subspaces of A^*.

Remarks:

(i) For a group algebra $L^1(G)$: G a lca group, CF and WCF are respectively the space of almost periodic functions and weakly almost periodic functions.

(ii) For the Eymard algebra $A_2(G)$, G a lc group, the spaces CF and WCF are again called as the space of almost periodic and weakly almost periodic functionals by Dunkl and Ramirez; Granirer [20]. Notice that if G is abelian, then $A_2(G)$ coincides with $L^1(\Gamma)$ where Γ is the dual group of G. We know that for $L^1(G)$, G-abelian, $L^1(G)^{**}$ is commutative if and only if G is finite (Cf. Civin and Yood [5, Theorem 3.14]). But at the same time, for any simisimple commutative Banach algebra the Birtel algebra (5.1.5 (3)) is always commutative B-algebra extending the multiplication of A. Therefore it is always interesting to know how far the commutativity of A can be extended among the quotient algebras. The following discussions centre around this problem.

5.2.5 Remark:

Let X be a introverted subspace of A^*. It can be easily seen that

(i) The transpose of $a \to fa$ is $F \to Ff$ for a fixed f in X.

(ii) The transpose of $f \to Ff$ is $G \to G.F$ for a fixed F in B_X.

(iii) for any $F \varepsilon B_X$ the right multiplication operator $\rho(F)$ is weak * - weak * continuous from B_X into B_X, since it is the transpose of the map $f \to Ff$ from X into X.

In view of the above remark the following lemma will not be surprising.

5.2.6 Lemma:

Let A be a commutative Banach algebra. Let X be a (norm closed) introverted subspace and F be in B. Then F is in the centre of B iff the left multiplication operator $\lambda(F): B \to B$ is weak $*$- weak $*$ continuous.

Proof:

If F is in the centre of B, then $\lambda(F) = \rho(F)$ and is weak $*$ -weak$*$ continuous by remark (ii) above.

Conversely assume that $\lambda(F)$ is weak $*$- weak $*$ continuous. Since A is commutative, we can easily see that,

$$a \cdot G = G \cdot a, \quad \forall\, a \varepsilon A, \quad \forall\, G \varepsilon B$$

Let $G \varepsilon B$. Let $\{a_i\} \subseteq A$ be such that, $\{a_i\}$ converges to G in weak-$*$ topology. Hence $F \cdot a_i \to F \cdot G$ and $a_i \cdot F \to G \cdot F$ as $\lambda(F)$ and $\rho(F)$ are weak $*$ - weak $*$ continuous. Since $a_i \cdot F = F \cdot a_i$ for each i, we conclude that $F \cdot G = G \cdot F$. This completes the proof.

5.2.7 Theorem:

Let A be a commutative B-algebra, X be a (norm closed) introverted subspace of A^*. Then the following are equivalent.

(i) B is commutative.

(ii) for every $F \varepsilon B$, $\lambda(F): B \to B$ is weak $*$ - weak $*$ continuous.

(iii) for every $f \varepsilon X$, the map $F \to Ff$ from B into X is weak $*$ - weak continuous.

(iv) $X \subseteq WCF$, the space of all weakly compact functionals.

Proof:

By lemma 5.2.6 (i) and (ii) are equivalent.

It is easy to see that (ii) and (iii) are equivalent.

To prove (iii) and (iv) are equivalent, we use the fact that for any two B-spaces X and Y, an operator $T: X \to Y$ is weakly compact iff $T^*: Y^* \to X^*$ is weak * - weak continuous (see Dunford and Schwartz [14, VI, 4.5]).

Now let $f \in WCF$. Then the operator $a \to fa$ is weakly compact i.e. the transpose of it is weak * - weak continuous which is precisely the map defined in (iii) (see 5.2.5 (i)). By retracing the steps, we see that the converse is also true.

Hence the theorem.

Remark:

The statement that (i) and (iv) are equivalent in the above theorem is the theorem 5.6 of A.T. Lau [32] for the Banach algebra $A_2(G)$.

Finally, we end this section, by proving the following theorem which was proved by Granirer [20, Proposition 2, p. 374] for the case $A = A_2(G)$.

5.2.8 Theorem:

Let A be a commutative semisimple regular, Tauberian B-algebra. If $\Delta(A)$ is discrete then,

$$\langle A^*A\rangle \subseteq CF \subseteq WCF.$$

<u>Proof</u>:

The only non-trivial part in the theorem is to show that for every $f \varepsilon A^*$, $a \varepsilon A$, $fa \ \varepsilon$ CF.

Because of the hypotheses on A, for every $\tau \ \varepsilon \ \Delta(A)$ there exists a_τ in A such that $\hat{a}_\tau = \chi_{\{\tau\}}$ and the space of finite linear sums of the form $\sum_{i=1}^{n} \alpha_i \, a_{\tau_i}$ is a dense subset of A.

Thus it is sufficient to prove that for any $f \varepsilon A^*$, $a_\tau \varepsilon A$, $fa_\tau \varepsilon$ CF. Let $g = fa$.

For any $b \varepsilon A$, $(fa_\tau)b = f(a_\tau b) = \tau(b) fa_\tau = \tau(b)g$. Therefore the set $\{gb: \ \|b\| \leq 1\}$ is a bounded subset of the 1-dimensional subspace spanned by g. Therefore $g \ \varepsilon$ CF and so $\langle A^* A\rangle \subseteq$ CF.

Hence the theorem.

<u>Remark</u>:

We give an example of a commutative semisimple, regular, Tauberian B-algebra such that $\langle A^*A\rangle \subseteq$ WCF but $\Delta(A)$ is not discrete, which shows that the converse of the Theorem 5.2.8 is not true.

Let $A = C(K)$, where K is an infinite compact T_2 space. Since A has a unit, $\langle A^*A\rangle = A^* = M(K)$, the space of finite regular Borel measures on K.

<u>Claim</u>: $A \subseteq$ WCF.

Let $\mu \ \varepsilon$ A be positive. Consider the map $T_\mu : A \to A$ defined by $T_\mu(f) = \mu f$. Let S denote the unit ball in C(K).

We prove that T_μ (S) is relatively weakly compact for which it is sufficient to prove that T_μ (S) is a relatively compact subset of $L^1(\mu)$.

Towards this, it is enough to check the following, because of Dunford and Pettis theorem (Cf. Diestel [10])

(a) $\sup_{f \varepsilon S} \|f d\mu\| \leq \infty$.

(b) $\sup_{f \varepsilon S} \int_E |f| \, d\mu \leq \mu(E)$ for every Borel set E.

But both are trivially true in this case. Therefore T_μ (S) is relatively weakly compact for every positive measure μ and thus for every measure μ on K.

Hence, $A^* \subseteq$ WCF.

But $\Delta(A) = K$ is not discrete.

<u>Remark</u>:

But for $A_2(G)$, the converse is also true (Cf. Granirer [20, proposition 2]).

<u>Section 5.3</u>:

In this section we intend to study the introverted subspaces of A^*, where A is not necessarily commutative but has a 1-right approximate identity (Cf. 1.2.1). Let X be a norm closed total left introverted subspace of A^* and let $B = B_X = A^{**}/X^\perp$. We imbed the algebra of left multipliers of A in B and give a characterization for any element in B to be a left multiplier of A. The main theorem in this section gives a characterization

for the operators from X into X commuting with actions of A. We discuss some of its consequences also.

With this assumption on A, it is well known that, A^{**} will have a right unit. In fact, by Banach Alaoglu's theorem, there exists a subnet of $\{e_i\}$ which we again denote by $\{e_i\}$ and $E \varepsilon A^{**}$ such that $\{e_i\}$ converges to E in weak-* topology. (Here $\{e_i\}$ denotes the 1-right approximate identity).

It is now easy to see that $Ef = f$ for every $f \varepsilon X$ and $G \cdot E = E$ for every G in A^{**}.

<u>5.3.1</u> Proposition:

Let X be a left introverted subspace of A^*. Then B has a right unit.

<u>Proof</u>:

Since $B = \dfrac{A^{**}}{X^\perp}$, the image of E in B will serve as a right unit.

We denote this unit by E_B or simply by E when there is no confusion.

Let X be a left introverted subspace of A^*. We denote the algebra of bounded linear operators on X such that $T(fa) = (Tf)a, \ \forall f \varepsilon X, \ a \varepsilon A$ by $L_A(X)$.

<u>5.3.2</u> Proposition:

Let X be a left introverted subspace of A^*. Then the following are true.

(i) $M_{\ell}(A)$, the algebra of multipliers on A is anti-isomorphically imbedded in $L_A(X)$.

(ii) $L_A(X)$ is isomorphically imbedded in B.

Both these imbeddings are norm-decreasing.

Proof:

We show that for every $T \varepsilon M_{\ell}(A)$, X is invariant under T^*.

Let $f\varepsilon X$. Now $T^{**}(E)$ is in A^{**}. We prove that $T^*(f)$ belongs to X by showing $T^*(f) = T^{**}(E)f$. Observe that

$T^*(fa) = T^*(f)a, \ \forall f\varepsilon X$ and $\forall \ a\varepsilon A$. Now,

$$(a, T^{**}(E)f) = (T^*(fa), E) = (T^*(f)a, E)$$

$$= (T^*(f), a\cdot E) = (T^*(f), a).$$

The map $T \to T^*$ is the anti-isomorphic imbedding required in (i).

Let $S \varepsilon L_A(X)$ then since $E_B f = f$ for every $f\varepsilon X$, we get that $S^*(E_B)f = S(f)$ for every $f\varepsilon X$.

Consider the map from $L_A(X)$ into B defined by $S \to S^*(E_B)$. Then it is easy to see that this map is an algebra isomorphism.

It is trivial to check that the above mappings are norm decreasing also.

5.3.3 Remarks:

(i) The map $S \to S^*(E_B)$ defined in (ii) of the above proposition satisfy the following. If $a\varepsilon A$, $f\varepsilon X$ then,

$$(a, Sf) = (fa, S^*(E_B))$$

for, $(fa, S^*(E_B)) = (S(fa), E_B) = ((Sf)a, E_B)$

$$= \lim_i ((Sf)a, e_i) = \lim_i (ae_i, Sf) = (a, Sf)$$

(ii) By the above proposition, there exists a linear norm decreasing anti-isomorphism from $M_\ell(A)$ into B which maps every T in $M_\ell(A)$ into $T^{**}(E_B)$. Furthermore, for every $T \varepsilon M_\ell(A)$ and $a\varepsilon A$, we have, $a \cdot T^{**}(E_B) = T(a)$.

5.3.4 Theorem:

Let X be an introverted subspace of A^*. Let $i: M_\ell(A)$ into B be the imbedding as in proposition 5.3.2. Then the following are equivalent. Let $F\varepsilon B$.

(i) $F \varepsilon i(M_\ell(A))$.

(ii) For every $a\varepsilon A$, the bounded linear functional $a \cdot F$ on X is $\sigma(X,A)$ - continuous.

(iii) The map $f \to Ff$ is weak* - weak* continuous.

(iv) For every $a\varepsilon A$, $a \cdot F \varepsilon i(A)$.

Proof:

Since $F \varepsilon i(M_\ell(A))$, there exists a $T \varepsilon M_\ell(A)$ such that $T^{**}(E_B) = F$, and so $a \cdot F = T(A)$, $\forall\ a\varepsilon A$ by the above remark.

Hence (i) implies (ii).

(ii) $\Rightarrow$ (iii) is easy to see.

Since the map $f \to Ff$ is weak* -weak* continuous, we have for every $a\varepsilon A$, there exist an element b such that

$$(b,f) = (a, Ff) = (f, a \cdot F) \text{ i.e. } a \cdot F \varepsilon i(A), \forall\ a\varepsilon A.$$

Hence (iii) $\Rightarrow$ (iv).

Define $T: A \to A$ by $i(T(a)) = a \cdot F$.

Since X is total, it is well defined, and the mapping is linear. It can be easily seen that $(f, a \cdot T(b)) = (f, T(ab))$ $\forall\ f \varepsilon X$ and $a, b \ \varepsilon\ A$.

Thus $T \ \varepsilon\ M_{\ell}(A)$ and therefore (iv) $\Rightarrow$ (i).

Hence the theorem.

Remark:

In this remark, let us improve the above theorem with Birtel's result in this context for a commutative, semisimple regular Tauberian Banach algebra.

We say that a function φ on $\Delta(A)$ belongs locally to $\hat{A}$ at each τ in $\Delta(A)$, if there exists a neighbourhood U of τ and a in A such that $\varphi/U = \hat{a}/U$ and at infinity if there exists a compact set K and $a \varepsilon A$ such that $\varphi/K^c = \hat{a}/K^c$. It is well known that a function φ on $\Delta(A)$ belongs locally to $\hat{A}$ at each $\tau\ \varepsilon\ \Delta(A)$ and at infinity then φ actually belongs to $\hat{A}$, (See Loomis [33, p. 85]).

For a commutative regular semisimple Tauberian B-algebras we have an improved version of the theorem 5.3.4.

5.3.5 Theorem:

Let A be a commutative regular, semisimple Tauberian B-algebra with bounded approximate identity. Let $F\ \varepsilon\ B$. Then all the four conditions in Theorem 5.3.4 are equivalent to

(v). The function $F/\Delta(A)$ belongs locally to $\hat{A}$ at each point of $\Delta(A)$.

Proof:

We prove (iv) and (v) are equivalent. Assume (iv). Let $\tau\varepsilon\Delta(A)$. If U is a compact neighbourhood of τ then there exists an element $a\varepsilon A$ such that $\hat{a}/K = 1$ and since $a\cdot F \in A$, we obtain (v), for, $(\omega, a\cdot F) = \hat{a}(\omega)F(\omega)$.

Now we assume (v). We observe that for every $a\varepsilon A$ such that $\hat{a}$ has compact support, the hypothesis implies that $a\cdot F$ belongs locally to A at every $\tau \in \Delta(A)$ and at infinity.

Therefore, $(a\cdot F)(\omega) = \hat{b}(\omega)$ for some $b\varepsilon A$.

So, $a\cdot F \in i(A)$. Use Tauberianness to conclude that $a\cdot F$ belongs to $i(A)$ for any arbitrary $a\varepsilon A$. Hence (iv).

The following theorem gives a characterization for $L_A(X)$.

5.3.6 Theorem:

Let A be a Banach algebra with 1- right approximate identity. Let X be a left introverted subspace of A^*. Then $L_A(X)$ is isometrically isomorphic with Y^* where Y is the subspace spanned by $\{fa;\ f\varepsilon X;\ a\varepsilon A\}$.

Proof:

Observe that by 5.1.3 (1) Y is also a left introverted subspace of A^* and so, Y^* is a Banach algebra.

Define $P: Y^* \to L_A(X)$ by

$P(F)(f) = Ff, \ \forall\, F\varepsilon Y;\ f\varepsilon X.$

Then for every F in Y^*, $P(F)$ is linear and, if $a \varepsilon A$, $f \varepsilon X$ we have, $P(F)(fa) = F(fa) = (Ff)a = (P(F)(f)a$ and, $\|P(F)(f)\| \leq \|F\| \, \|f\|$. Therefofe $P(F)$ belongs to $L_A(X)$.

It can be easily seen that $\|P(F)\| \leq \|F\| \; \forall F \varepsilon Y$ and P is an algebra homomorphism.

We prove that P is onto. Let $T \varepsilon L_A(X)$. By remark 5.3.3 (i), there exists a $F \varepsilon B = X^*$ satisfying

$(fa, F) = (a, T(f))$. But $F \varepsilon Y^*$ also. Therefore P is onto.

It remains to show that P is an isometry. Since A has 1-right approximate identity, and the norm closure of the subspace Y is the non-degenerate (essential) part of X when X is considered to be a representation space of A, we can apply Hewitt-Cohen's Factorization theorem to Y (Cf. Hewitt-and Ross [28]).

Now let $g \varepsilon Y$, $\|g\| \leq 1$ and $\varepsilon > 0$. Then, there exists an $f \varepsilon X$ and $a \varepsilon A$ such that $\|a\| \leq 1$, $\|f-g\| < \varepsilon$ and $g = fa$. If $F \varepsilon Y^*$ then,

$$|(g,F)| = |(fa, F)| = |(a, Ff)| = |(a, P(F)f)|$$

$\leq \|P(F)f\| \leq \|P(F)\|(1+\varepsilon)$. Since ε is arbitrary, we have, $\|F\| \leq \|P(F)\|$ and therefore P is an isometry.

Hence the theorem.

We shall give some corollaries of the above theorem, and discuss some examples.

5.3.7 Corollary:

If X is further assumed to be non-degenerate in the above theorem, then we have $L_A(X) \cong B_X$ (= X^*).

5.3.8 Corollary:

Let A be as in the theorem 5.3.6, and X is $\langle A\ A\rangle^-$ (Cf. example (1) of 5.1.3), then $L_A(A^*)$ is isometrically isomorphic to $B_X = X^*$.

We further observe that since the norm closure of $\langle A^*A\rangle = A^*A$ by Hewitt-Cohen's factorization theorem, an introverted space X is non-degenerate iff X is contained in the norm closure of $\langle A^*A\rangle$. Let us give some examples of introverted subspaces contained in $\langle A^*A\rangle^-$.

Let A be commutative. Then WCF $\subseteq \langle A^*A\rangle^-$. (Refer 5.2.2 for the definition).

Suppose that f ε WCF. We show that f belongs to the weak closure of the subspace $\langle A^*A\rangle$. Consider the net $\{fe_i\}$ in the orbit O(f) of f. Take a subnet if necessary, which we again denote by $\{fe_i\}$ and a $g \varepsilon A$, such that $\{fe_i\}$ converges to g weakly.

But, notice that $\{fe_i\}$ converges to f in the weak -* topology. Therefore $\{fe_i\}$ converges to f weakly. Since $\{fe_i\} \subseteq A^*A$, $f \ \varepsilon \ \langle A^*A\rangle^-$.

Therefore, the norm closure of $[\Delta(A)]$; CF, WCF are all contained in $\langle A^*A\rangle^-$.

Finally, we discuss some classical examples.

(i) If $A = L^1(G)$, G a lc group, then,

$$L_{L^1(G)}(L^\infty(G)) = (C_{ru}(G))^*,$$

where $C_{ru}(G)$ denotes the space of all bounded right uniformly continuous functions on G.

(ii) Let G be abelian, and AP(G) be the algebra of all almost periodic functions. Then the space of all bounded linear operators from AP(G) into AP(G) commuting with left convolution operators is precisely $M(\beta(G))$ where $\beta(G)$ denotes the Bohr compactification of G. Notice that $AP(G) = ([\Gamma])^-$ where Γ is the dual group and $M(\beta(G)) = (AP(G))^*$.

Remark:

A.T. Lau proved the theorem 5.3.6 for $A = L^1(G)$ in [31, Theorem 1] and then separately for $A = A_2(G)$ in [32], G an amenable group. Corollary 5.3.7 was proved by Grosser and Losert [24, Section 5] for arbitrary Banach algebra with 1-right approximate identity.

Section 5.4:

This section is for some of the applications of the results, we established in earlier sections of this chapter and Chapter 4.

Let us assume that A is commutative M-regular N algebra, for some $M > 0$. This assumption is valid throughout this section and M is also fixed.

Let X be any introverted subspace containing $\Delta(A)$. For any $\omega \in \Delta(A)$, we denote the set $\{F \in B: F(\omega) = 1$ and $a \cdot F = \omega(a)F$ for every $a \in A\}$, where B is the dual X^*, by M_ω.

5.4.1 Proposition:

Let X be a introverted subspace containing $\Delta(A)$. Then for every $\omega \in \Delta(A)$, M_ω is a non-empty convex set.

Proof:

The only non-trivial part to prove is that M_ω is not empty.

By lemmas 4.1.5 and 4.1.6, there exists a net $\{a_i\}$ in A and a F in A^{**} satisfying the following.

$||a_i|| \leq M$ and $\omega(a_i) = 1 \quad \forall\ i$, $\{a_i\} \to F$ in the weak-$*$topology

Again, $(fa, F) = \omega(a)(f,F)$. In particular, $F/\Delta(A) = \chi_{\{\omega\}}$.

Since $F(\omega) = 1$ and $\omega \in X$, we have that $F \in B$ and since, $(f, a \cdot F) = (Ff, a) = (fa, F) = \omega(a)(f, F)$ we infer that F belongs to M_ω.

Hence the proposition.

Remark:

For the sake of definiteness, we denote the F we constructed in the above proposition by F^o.

5.4.2 Proposition:

Let A be a commutative M-regular N algebra, X be the space of all weakly compact functionals in A^*, (Refer the definition 5.2.2). Then $M_\omega = \{F^o\}$.

Proof:

Let G be in M_ω. Let $\{a_i\}$ be the net as in the above proposition satisfying the conditions given therein.

Let f be in WCF. It is clear that $\{fa_i\}$ converges to $(f, F^0)\cdot\omega$ in the weak-$*$ topology. But since f ε WCF, take a subnet, if necessary, which we again denote by $\{a_i\}$ so that $\{fa_i\}$ converges to $(f, F^0)\omega$ weakly. Therefore,

$$\lim_i (fa_i, G) = (f, F^0)(\omega, G) = (f, F^0).$$

But, $$\lim_i (fa_i, G) = \lim_i (f, a_i\cdot G) = \lim_i \omega(a_i)(f,G) =$$

$$= (f, G), \quad \text{for, } \omega(a_i) = 1 \quad \forall\ i.$$

Therefore, $(f, F^0) = (f, G), \quad \forall \ f \ \varepsilon$ WCF.

Hence $G = F^0$. This completes the proof of the proposition.

5.4.3 Theorem:

Let A be a M-regular N algebra. Suppose that the maximal ideal space $\Delta(A)$ contains no isolated points. Then there exists no non-trivial weakly compact multiplier.

Proof:

Suppose that the T is a weakly compact multiplier of A. We prove that $T = 0$ by showing that $\hat{T} = 0$.

Actually we show that if $\hat{T}(\omega) \neq 0$ for some ω then ω is an isolated point by proving $\chi_{\{\omega\}}$ is a continuous function on $\Delta(A)$.

Consider the space WCF. Notice that $\Delta(A) \subseteq$ WCF. Let F^{o} be as in proposition 5.4.2.

<u>Claim</u>:

$$G = \frac{T^{**}(F^{o})}{\hat{T}(\omega)}$$ belongs to M_{ω}, where, M_{ω} is as in proposition 5.4.2.

$$G(\omega) = \frac{1}{\hat{T}(\omega)}(\omega, T^{**}(F^{o}) = \frac{(T^{*}(\omega), F^{o})}{\hat{T}(\omega)}$$

$$= \frac{(T(\omega)\omega, F^{o})}{\hat{T}(\omega)} = 1$$

(See remarks after 4.1.2 for a discussion about $\hat{T}$).

So, G belongs to B, $G(\omega) = 1$.

We prove that $a \cdot G = \omega(a)G$ in B. It is sufficient to prove $a \cdot T^{**}(F^{o}) = \omega(a)T^{**}(F^{o})$.

Let $f \in$ WCF, $a \in A$.

$$\begin{aligned}(f, a \cdot T^{**}(F^{o})) &= (fa, T^{**}(F^{o})) = (T^{*}(fa), F^{o}) \\ &= (T^{*}(f)a, F^{o}) \quad \text{for, } T \text{ is a multiplier} \\ &= (T^{*}(f), a \cdot F^{o}) \quad \text{for, } A \text{ is commutative} \\ &= \omega(a)(T^{*}(f), F^{o}) = \omega(a)(f, T^{**}(F^{o})).\end{aligned}$$

Hence the claim.

By proposition 5.4.2, we get that $\frac{T^{**}(F^{o})}{\hat{T}(\omega)} = F^{o}$.

Since T is weakly compact, $T^{**}(A^{**}) \subseteq A$, (Refer Dunford and Schwartz [14, V. 4.1]). In particular, we have that there exists an element $x \in A$ such that, $F^{o}(\tau) = \tau(x) \ \forall \tau \in \Delta(A)$.

But we know that $F^o/\Delta(A) = \chi_{\{\omega\}}$. Therefore $\hat{x} = \chi_{\{\omega\}}$. Thus $\chi_{\{\omega\}}$ is a continuous function on $\Delta(A)$.

Hence the theorem.

Remarks:

(i) On the contrary, if ω is an isolated point, then there exists a non-trivial weakly compact multiplier.

By Shilov's idempotent theorem, there exists $a \varepsilon A$ such that $\hat{a} = \chi_{\{\omega\}}$ i.e., for every $b \varepsilon A$, $ab = \omega(b)a$. Therefore the left multiplication operator $\lambda(a)$ is then an operator of rank 1.

(ii) The above theorem is a classical result for $L^1(G)$: G a lca group. A.T. Lau [32, proposition 6.9] proved the same result for $A_2(G)$.

Finally, letus end this section by proving a theorem which gives various necessary and sufficient conditions for A to have a discrete maximal ideal space.

5.4.4 Theorem:

Let A be M-regular N algebra. Then the following are equivalent.

(i) The maximal ideal space, $\Delta(A)$ is discrete.

(ii) For every $a \varepsilon A$, the left multiplication operator $\lambda(a): A \to A$ is compact.

(iii) For every $a \varepsilon A$, $\lambda(a)$ is weakly compact.

(iv) A is isometrically imbedded as a two sided ideal in A^{**}

(v) $L_A(A^*) \cong M(A)$.

Proof:

In view of the theorem 4.1.8, it is sufficient to prove the equivalence of first four statements.

(i) => (ii):

Since $\Delta(A)$ is discrete, using the fact that, A is semisimple regular and Tauberian, we see that A is norm closure of finite linear sums of the form $\sum_{i=1}^{n} \alpha_i a_{\tau_i}$ where for every $\tau \varepsilon \Delta(A)$ a_τ is the unique element in A defined by $a_\tau = \chi_{\{\tau\}}$ and $\alpha_i \varepsilon \mathbb{C}$. But since, $a\ b = \tau(b)a$ for every $b\varepsilon A$, $\lambda(a\ )$ is infact a rank 1 operator, and hence $\lambda(a)$ is compact for every $a\varepsilon A$.

(iii => (iv):

Let $a\varepsilon A$.

Since $\lambda(a)$ is weakly compact, we have,

$(\lambda(a))^{**}(A^{**}) \subseteq A$ (Refer Dunford and Schwartz [14]).

But $(\lambda(a))^{**}(F) = a\cdot F$. Therefore, A is a left ideal in A^{**}. Since $a\cdot F = F\cdot a$, A is infact a two sided ideal in A^{**}.

(iv) => (i):

Let $\omega \varepsilon \Delta(A)$. Let F^o be as in proposition 5.4.2. Then, for any $a\varepsilon A$, $a\cdot F^o = \omega(a)\ F^o$.

Therefore, by hypothesis $F^o \varepsilon A$. But $F^o/\Delta(A) = \chi_{\{\omega\}}$.

So $\chi_{\{\omega\}}$ is continuous on $\Delta(A)$ for every $\omega\varepsilon \Delta(A)$.

Thus $\Delta(A)$ is discrete.

Hence the theorem.

<u>Remark</u>:

It is worth while to mention that the abelian version of the following theorem proved by Grosser [23] is contained in the above theorem.

$L^1(G)$: G a lc group, is an ideal in its bidual space iff G is compact.

CHAPTER 6

WEAK CONTAINMENT AMONG THE B-SPACE REPRESENTATIONS

In Section 6.1 we define Banach algebra lattices, give some examples and discuss lattice representations of a Banach algebra lattice. In Section 6.2, we study the notion of weak containment among the B-space representations of a B-algebra. This section contains the main theorem of this chapter, which was originally proved by Cowling and Fendler [8]. We give a simpler and modified proof, for this theorem. Motivated by the construction of Fourier-Stiltjes algebra for a locally compact group G due to Eymard, we construct a host of function algebras on G, in Section 6.3, using the results we obtained in the earlier sections.

<u>Section 6.1</u>:

<u>6.1.1</u> Definition:

We say that A is a Banach algebra lattice if,

(i) A is a Banach algebra.

(ii) A is a Banach lattice.

(iii) $|xy| \leq |x|\cdot|y|$, for every x,y in A.

As usual we denote the positive cone of A by A_+.

Examples:

(i) $C_o(S)$: S any locally compact T_2-space.

(ii) $L^\infty(S)$: (S,Σ,m) any measure space.

(iii) $L^1(G)$; $M(G)$: G any locally compact group.

(iv) $L^p(G)$; $1 \leq p < \infty$: G a compact group.

6.1.2 Definition:

Let A be a Banach algebra lattice, (π, X) a B-space representation. Then we say that π is a lattice representation if,

(i) X is a Banach lattice.

(ii) If $x \in A_+$ then $\pi(x)$ is a positive operator.

(See 1.1.3 for a discussion of B-lattices).

In particular if X is $L^p(S,\Sigma,m)$, $1 \leq p \leq \infty$, for some measure space (S,Σ,m), then we say that (π,X) is a p-representation.

Examples:

(i) The left regular representation (λ,A) of any Banach algebra lattice A is a lattice representation.

(ii) If $A = L^1(G)$: G a lc group, and $X = L^p(G)$, $1 \leq p < \infty$ then, $\lambda_p(f)(g) = f*g$; $f \in L^1(G)$, $g \in L^p(G)$, defines a p-representation.

Remarks:

Let A be a Banach algebra lattice.

(i) If (π,X) is a lattice representation, then the contragradient representation $(\bar{\pi},X^*)$ is also a lattice representation for the reverse algebra $\tilde{A}$ (Cf. 1.2.6).

(ii) If $\{(\pi_i,X_i)\}_{i\varepsilon I}$ is a family of lattice representations, then $(\sum_{i\varepsilon I}^{\oplus}\pi_i, \ell^p(I,X_i))$ is also a lattice representation. Moreover, if (π_i,X_i) is a p-representation for every i, then so is $(\sum_{i\varepsilon I}^{\oplus}\pi_i, \ell^p(I,X_i))$. (Cf. 1.1.2 and 1.2.5 for notations)

(iii) If (π,X) is a lattice (or p) representation, then so is its non-standard hull. (Cf. Section 2.3 for the definition).

Section 6.2:

Let A be a Banach algebra. We recall our convention that every B-space representation satisfies $\|\pi(x)\| \leq \|x\| \; \forall \; x\varepsilon A$ (Cf.1.2.4) If $\mathcal{R}$ is a collection of B-space representations, let $W_{\mathcal{R}}$ denote the following subspace of A^*.

$$\{f\varepsilon A^*: \exists \text{ a } K>0 \text{ such that } |f(a)| \leq K \operatorname*{Sup}_{\pi\varepsilon} \|\pi(a)\| \; \forall \; a\varepsilon A\} \qquad (*)$$

$$\text{Denote } \|f\|_{\mathcal{R}} = \operatorname{Inf}\{K>0: K \text{ satisfies } *\} \; \forall \; f\varepsilon W_{\mathcal{R}}$$

Remark:

If $\mathcal{R} = \{\pi\}$ then for any $f\varepsilon W$, we denote $\|f\|_{\mathcal{R}}$ by $\|f\|_{\pi}$. Observe that W_{π} with the above norm is precisely the dual of the subspace spanned by $\pi(A)$ in $L(X)$.

6.2.1 Definition:

Let $\mathcal{R}$ be a collection of B-space representations and π is another B-space representation. Then we say that π is weakly contained in $\mathcal{R}$ if $\|\pi(a)\| \leq \operatorname*{Sup}_{\rho\varepsilon\mathcal{R}} \|\rho(a)\|$ for every $a\varepsilon A$.

Notation: $\pi \lesssim \mathcal{R}$.

6.2.2 Remarks:

(i) Notice that if $\rho \in \mathfrak{R}$ then $W_\rho \subseteq W_{\mathfrak{R}}$; $\|f\|_{\mathfrak{R}} \leq \|f\|_\rho$ $\forall$ $f \in W$.

(ii) If A is a C* algebra and π, $\mathfrak{R}$ are * representations of A then the above definition coincides with the fact that $\bigcap_{\rho \in \mathfrak{R}} \operatorname{Ker} \rho \subseteq \operatorname{Ker} \pi$. This is the original definition of weak containment among * representations of a C-* algebra given by Fell [17].

6.2.3 Definition:

We say that any two B-space representations π, ρ are quasi equivalent if $\pi \lessapprox \rho$ and $\rho \lessapprox \pi$.

Notation $\pi \sim \rho$.

6.2.4 Remarks:

(i) Let (π, X) be a B-space representation. Let I be any index set. $1 < p < \infty$. Set $X_i = X$ for every $i \in I$. Then $\pi \sim \pi^{(p)}$ (Cf. 1.2.5 for the definition of $\pi^{(p)}$). For,

$$\|\pi^{(p)}(x)\| = \sup_{i \in I} \|\pi_i(x)\| = \|\pi(x)\| \quad \forall \quad x \in A.$$

(ii) If $(\hat{\pi}, \hat{X})$ denotes non-standard hull of (π, X) then we have, $\|\hat{\pi}(x)\| = \|\pi(x)\|$ for every $x \in A$. Therefore, $\pi \sim \hat{\pi}$.

6.2.5 Proposition:

Let $\mathfrak{R}$ be a collection of B-space representations of A. Then

(i) $\sup_{\rho \in \mathfrak{R}} \|\rho(a)\| = \sup_{\|f\|_{\mathfrak{R}} \leq 1} |f(a)|$ $\forall$ $a \in A$. Furthermore,

(ii) If π is any other B-space representation then

π is weakly contained in $\mathcal{R}$ if and only if

$$W_\pi \subseteq W_{\mathcal{R}} \text{ and } \|f\|_{\mathcal{R}} \leq \|f\|_\pi \text{ for every } f \in W_\pi.$$

Proof:

(i) is evident if $\mathcal{R}$ consists of a single representation. To prove (i) for any collection, we need to show that

$$\sup_{\rho\in\mathcal{R}} \|\rho(a)\| \leq \sup_{\|f\|_{\mathcal{R}}\leq 1} |f(a)| \;\forall\, a\in A.$$

Let $a\in A$ and $\varepsilon > 0$. Then there exists a $\rho'\in\mathcal{R}$ such that

$$\sup_{\rho\in\mathcal{R}} \|\rho(a)\| \leq \|\rho'(a)\| + \varepsilon$$

$$= \sup_{\|f\|_{\rho'}\leq 1} |f(a)| + \varepsilon$$

$$\leq \sup_{\|f\|_{\mathcal{R}}\leq 1} |f(a)| + \varepsilon \quad \text{by remark (i) above.}$$

Since ε is arbitrary (i) follows.

Now we prove (ii).

'only if' part is just from the definition. We prove the other way. Let $a\in A$.

$$\|\pi(a)\| = \sup_{\|f\|_\pi\leq 1} |f(a)|$$

$$\leq \sup_{\|f\|_{\mathcal{R}}\leq 1} |f(a)| = \sup_{\rho\in\mathcal{R}} \|\rho(a)\| \quad \text{by (i).}$$

Hence the proposition.

6.2.7 Proposition:

Let $\mathcal{R}$ be a collection of B-space representations of A. Then there exists another B-space representation π of A such that $W_{\mathcal{R}} = W_\pi$.

Proof:

Let $1 < p < \infty$. Let $\{\pi_i, X_i\}_{i\varepsilon I}$ denote the collection $\mathcal{R}$. Let (π, X) denote the B-space representation,

$\left(\sum_{i\varepsilon I}^{\oplus} \pi_i, \ell^p(I, X_i)\right)$. Then observe that $\|\pi(a)\| = \sup_{i\varepsilon I} \|\pi_i(a)\|$ $\forall$ $a\varepsilon A$. Hence the proposition.

Let (π, X) be a B-space representation of A. Then there exists another subspace of A^* associated with π.

For $\xi \varepsilon X$, $\eta \varepsilon X^*$, we denote the bounded linear functional on A, $a \to (\pi(a)\xi, \eta)$ by $\pi_{\xi,\eta}$. These are called the coordinate functionals belonging to π.

Let T_π denote the subspace of all bounded linear functionals f which admits the representation of the form,

$$\sum_{n\varepsilon \mathbb{N}} \pi_{\xi_n, \eta_n} \quad \text{with} \quad \sum_{n\varepsilon \mathbb{N}} \|\xi_n\| \, \|\eta_n\| < \infty.$$

Remarks:

(i) We can easily observe the following.

(a) $T_\pi \subseteq W_\pi$ with, $\|f\|_\pi \leq \sum_{n\varepsilon \mathbb{N}} \|\xi_n\| \, \|\eta_n\|$

(b) $T\pi$ is contained in the set of all coordinate functionals belonging to the representation $(\pi^{(p)}, \ell^p(\mathbb{N}, X))$ for any p such that $1 < p < \infty$.

(ii) If A is a liminal C-* algebra then, for every irreducible * representation $(\pi, \mathcal{H})$, $T_\pi = W_\pi$.

Since A is liminal, $\pi(A)$ equals the algebra $LC(\mathcal{H})$ of all compact operators on $\mathcal{H}$ (Cf. Dixmier [11, Chapter 4]).

Thus W_π is the dual of $LC(\mathcal{H})$ which is precisely T_π.

But this is not the case with every B-space representation of a B-algebra A. For example, if $A = L^1(G)$, G a lc group and π the left regular representation, λ , then we can easily see that $W_\pi = L^\infty(G)$ whereas T_π is the space of bounded right uniformly continuous functions on G.

<u>6.2.8</u> Theorem:

Let A be a B-algebra and let (π, X) be a B-space representation. Then there exists another representation (π', X') such that

(i) $\pi \sim \pi'$

(ii) For every $f \varepsilon W_\pi$, there exist $\xi \varepsilon X'$; $\eta \varepsilon (X')^*$ such that $f = \pi'_{\xi,\eta}$ and $\| f \|_\pi = \| \xi \| \, \| \eta \|$.

<u>Remark</u>:

This theorem was originally proved by Cowling and Fendler [8]. We prove this theorem essentially following their ideas but using non-standard methods so that the proof becomes simpler.

For proving the theorem we need the following lemmas.

Let $1 < p < \infty$ be fixed, throughout this proof.

6.2.9 Lemma:

Let U denote, the unit ball of W_π. Then,

$$V = \text{co } \{\pi_{\xi,\eta} : \|\xi\| \leq 1; \|\eta\| \leq 1\}$$

is $\sigma(W_\pi, A)$ dense in U. (See 1.1.1 for the notation).

Proof:

Suppose that the assertion is not true. Then there exists a f in U such that $f \notin \overline{V}$. By Hahn-Banach theorem, there exists an element x in A such that $f(x) \neq 0$ but $f/\overline{V} = 0$, i.e., for every $\xi \in X$, $\eta \in X^*$ such that $\|\xi\| \leq 1$, $\|\eta\| \leq 1$, we have $\pi_{\xi,\eta}(x) = 0$. Therefore, $\pi(x) = 0$.

But since $|f(x)| \leq \|f\|_\pi \|\pi(x)\|$, we have $f(x) = 0$, which is a contradiction.

Hence the lemma.

6.2.10 Lemma:

Let $\varepsilon > 0$ and $\{a_1, \ldots, a_n\}$ be a finite set in A. Let $f \in W_\pi$ be such that $\|f\|_\pi \leq 1$. Then there exists a $\xi \in \ell^p(\mathbb{N}, X)$ and $\eta \in \ell^q(\mathbb{N}, X^*)$ such that

$$\|\xi\| \leq 1; \quad \|\eta\| \leq 1 \text{ and } |f(a_i) - \pi^{(p)}_{\xi,\eta}(a_i)| < \varepsilon \quad \forall\ i, 1 \leq i \leq n$$

Proof:

Let V be as in Lemma 6.2.9. Then any element g of V is of the form

$$\sum_{i=1}^{n} \alpha_i \pi_{\xi_i,\eta_i} \quad \text{where, } \sum_{1}^{n} \alpha_i = 1,\ 0<\alpha_i<1 \text{ and}$$

$$\|\xi_i\| \leq 1, \ \|\eta_i\| \leq 1 \quad \forall\ i.$$

Let, $\xi = (\alpha_1^{1/p} \cdot \xi_1, \ldots \alpha_n^{1/p} \cdot \xi_n, 0, 0, \ldots\ldots)$

and $\eta = (\alpha_1^{1/q} \cdot \eta_1, \ldots \alpha_n^{1/q} \cdot \eta_n, 0, 0, \ldots\ldots)$

Then observe that $g = \pi_{\xi,\eta}^{(p)}$ and

$\xi \in \ell^p(\mathbb{N},X)$ with $\|\xi\| \leq 1$; $\eta \in \ell^q(\mathbb{N},X^*)$ with $\|\eta\| \leq 1$.

Now the result follows from Lemma 6.2.9.

<u>6.2.11</u> Lemma:

Suppose that $a \in A$, $0 < \varepsilon < 1$ and $f \in W_\pi$ be such that $\|f\|_\pi = 1$. Then there exists $\tilde{\xi} \in \ell^p(\mathbb{N},X)$ and $\tilde{\eta} \in \ell^q(\mathbb{N},X^*)$ satisfying the following.

(i) $\|\tilde{\xi}\| = 1 = \|\tilde{\eta}\|$ and,

(ii) $|f(a) - \pi_{\tilde{\xi},\tilde{\eta}}^{(p)}(a)| < \varepsilon \cdot \|\pi(a)\|$.

<u>Proof</u>:

Choose a $\delta > 0$ such that $\delta + \frac{2}{1-2\delta} < \varepsilon$.

Since $\|f\|_\pi = 1$, there exists an element b in A such that

$$|f(b)| \geq (1-\delta)\ \|\pi(b)\|$$

Choose a ξ and η as in Lemma 6.2.10 so that $\|\xi\| \leq 1$; $\|\eta\| \leq 1$

and $|f(a) - \pi_{\xi,\eta}^{(p)}(a)| \leq \delta\|\pi(a)\|$;

$|f(b) - \pi_{\xi,\eta}^{(p)}(b)| \leq \delta\|\pi(b)\|$.

Now, we have, $1 \geq \|\xi\| \cdot \|\eta\| \geq \|\pi_{\xi,\eta}^{(p)}\|_{\pi(p)}$ (*)

But, $|\pi_{\xi,\eta}^{(p)}(b)| \geq |f(b)| - |f(b) - \pi_{\xi,\eta}^{(p)}(b)|$

$\geq (1-\delta)\ \|\pi(b)\| - \delta\ \|\pi(b)\|$

$= (1-2\delta)\ \|\pi(b)\|$

Since $\|\pi^{(p)}(b)\| = \|\pi(b)\|$, we have, $\|\pi^{(p)}_{\xi,\eta}\|_{\pi(b)} \geq (1-2\delta)$.

Using $(*)$, we conclude that $1 \geq \|\xi\|\,\|\eta\| \geq 1-2\delta$. $\qquad (**)$

Take $\tilde{\xi} = \xi/\|\xi\|$ and $\tilde{\eta} = \eta/\|\eta\|$. Then we have

$$|f(a) - \pi^{(p)}_{\tilde{\xi},\tilde{\eta}}(a)| \leq |f(a) - \pi^{(p)}_{\xi,\eta}(a)| + |\pi^{(p)}_{\tilde{\xi},\tilde{\eta}}(a) - \pi^{(p)}_{\xi,\eta}(a)|$$

$$\leq \delta\|\pi(a)\| + \{|1 - \frac{1}{\|\xi\|\,\|\eta\|}\}\|\pi(a)\|$$

But, $\left|1- \frac{1}{\|\xi\|\,\|\eta\|}\right| = \frac{1-\|\xi\|\,\|\eta\|}{\|\xi\|\,\|\eta\|} \leq \frac{2\delta}{1-2\delta}$ by $(**)$.

Therefore, $|f(a) - \pi^{(p)}_{\tilde{\xi},\tilde{\eta}}(a)| < (\delta + \frac{2\delta}{1-2\delta})\,\|\pi(a)\|$

$< \varepsilon\,\|\pi(a)\|$ as we required.

Hence the lemma.

Remark:

Summarizing the above, we proved that for every $f \in W_\pi$, there exist nets $\{\xi_i\}_{i\in I}$ in $\ell^p(\mathbb{N},X)$ and $\{\eta_i\}$ in $\ell^q(\mathbb{N},X^*)$ such that

(i) $\|\xi_i\| = 1$, $\|\eta_i\| = \|f\|_\pi$ $\forall$ i and

(ii) $\lim_i |f(a) - \pi^{(p)}_{\xi_i,\eta_i}(a)| = 0$, $\forall$ $a\in A$.

Now let us prove theorem 6.2.8.

Let $f \in W_\pi$. For the sake of simplicity we denote the representation $(\pi^{(p)}, \ell^p(\mathbb{N},X)$ by (ρ, Y). Let $(\hat{\rho}, \hat{Y})$ be non-standard hull of (ρ, Y) (Cf. Section 2.3).

By Transfer principle (See 1.3.1) applied to the nets defined in the above remark, there exist nets $\{\xi_i\}_{i\in {}^*I}$, in ${}^*Y$ and

$\{\eta_i\}_{i\varepsilon{}^*I}$ in ${}^*(Y^*)$ such that,

$\|\xi_i\| = 1; \ \|\eta_i\| = \|f\|_\pi \quad \forall\, i \,\varepsilon\, {}^*I.$

In fact, we observe that $\xi_i \,\varepsilon\, \hat{Y}$ and $\eta_i \,\varepsilon\, (Y^*)^\wedge \subseteq (\hat{Y})^*$ $\forall$ i.

Now let $a\varepsilon A$, $\varepsilon > 0$.

Then there exists $k \,\varepsilon\, I$ such that

$$\models (\forall\, i\varepsilon I)(i > k \Rightarrow |f(a) - \rho_{\xi_i,\eta_i}(a)| < \varepsilon).$$

By transfer principle,

$$* \models (\forall\, i \,\varepsilon\, {}^*I)\ (i > k \Rightarrow |f(a) - \hat{\rho}_{\xi_i,\eta_i}(a)| < \varepsilon\ ) \qquad (\dagger)$$

(Notice that ε and k are standard individuals).

But by concurrence theorem (see 1.3.2 or Robinson [37,p.97]) there exists $j \,\varepsilon\, {}^*I$ such that $j > i$ $\forall$ $i\varepsilon I$.

Therefore by $(\dagger)$, we have, $\forall$ $a\varepsilon A$, $\forall$ $\varepsilon > 0$,

$$|f(a) - \hat{\rho}_{\xi_j,\eta_j}(a)| < \varepsilon.$$

Thus, $|f(a) - \hat{\rho}_{\xi_j,\eta_j}(a)| \approx 0$, $\forall$ $a\varepsilon A$.

Since, $|f(a) - \hat{\rho}_{\xi_j,\eta_j}(a)|$ is a standard real number,

we have, $|f(a) - \hat{\rho}_{\xi_j,\eta_j}(a)| = 0$.

Therefore, $f(a) = \hat{\rho}_{\xi_j,\eta_j}(a)$ $\forall$ $a\varepsilon A$.

We recall that $\rho = \pi^{(p)}$. Again by remarks 6.2.4 (i) and (ii) we have $\pi \sim \pi^{(p)}$ and $\rho \sim \hat{\rho}$.

Put $\hat{\rho} = \pi'$; $\xi_j = \xi$, $\eta_j = \eta$.

Thus, we observe that $\pi \sim \pi'$ and $f = \pi'_{\xi,\eta}$ with $\|f\|_\pi = \|\xi\|\,\|\eta\|$.

Hence the theorem.

<u>6.2.12</u> Corollary:

Let A be a Banach algebra lattice, If (π,X) is a lattice representation, then so is (π', X') and if (π,X) is a p-representation for some fixed p, $1 < p < \infty$, then (π', X') is also a p-representation.

<u>Remark</u>:

Let A be a B-algebra (π,X) be a non-degenerate B-space representation. Since we know that non-standard hull of a non-degenerate representation need not be non-degenerate, we cannot expect to have the resulting representation (π', X') in the above theorem to be non-degenerate. But we can still achieve this in some special cases.

We shall need the following proposition due to Cowling and Fendler [8]. We state it without proof.

<u>6.2.13</u> Proposition:

Let A be a B-algebra with 1-approximate identity. Let (π,X) be a B-space representation with X being reflexive. Then there exists a projection P with $\|P\| = 1$ in $L(X)$ such that

$$P \circ \pi(a) = \pi(a) \circ P \qquad \forall\ a \in A$$

$$\text{Range of } P = [\pi(A)X]^-$$

$$\text{Kernel of } P = \{\xi \in X : \pi(a)\xi = 0,\ \forall\ a \in A\}.$$

6.2.14 Proposition:

Let A be a B-algebra with 1 approximate identity $\{e_i\}$. Let (π, X) be a B-space representation. Let $Y = [\pi(A)X]^-$ and $\rho = \pi/Y$. Then the sub-representation ρ is quasi-equivalent to π.

Proof:

We shall prove that $\|\pi(a)\| = \|\rho(a)\| \;\; \forall\, a \varepsilon A$.
But it is sufficient to show that $\|\pi(a)\| \leq \|\rho(a)\| \;\; \forall\, a \varepsilon A$.

Let $a \varepsilon A$; $\varepsilon > 0$ and $\xi \varepsilon X$.

$$\begin{aligned} \|\pi(ae_i)\xi\| &= \|\rho(a)\;(\pi(e_i)\xi)\| \\ &\leq \|\rho(a)\|\;\|\pi(e_i)\xi\| \\ &\leq \|\rho(a)\|\;\|\xi\|, \text{ since } \|e_i\| \leq 1 \text{ and } \|\pi(x)\| \leq \|x\| \\ &\qquad\qquad\qquad\qquad \forall\, x \varepsilon A. \end{aligned}$$

Hence, $\|\pi(ae_i)\| \leq \|\rho(a)\|$.

But since $\|\pi(ae_i) - \pi(a)\| \to 0$, we have $\|\pi(a)\| = \lim\limits_i \|\pi(ae_i)\| \leq \|\rho(a)\|$.

Hence the proposition.

6.2.15 Corollary:

Let A be a B-algebra with 1-approximate identity $\{e_i\}$. Let (π, X) be a non-degenerate B-space representation with X, a reflexive space. Then there exists a non-degenerate B-space representation (π', X') such that,

(i) $\pi \sim \pi'$

(ii) $\forall\, f \varepsilon W_\pi$, there exists $\xi \varepsilon X'$; $\eta \varepsilon (X')^*$ such that $\|f\|_\pi = \|\xi\|\;\|\eta\|$ and $f = \pi'_{\xi,\eta}$.

Proof:

Let (ρ, Y) be a B-space representation such that $\pi \sim \rho$ and $f = \rho_{\tilde{\xi},\tilde{\eta}}$ for some $\tilde{\xi} \varepsilon Y$, $\tilde{\eta} \varepsilon Y^*$ and $\|f\|_\pi = \|\tilde{\xi}\| \, \|\tilde{\eta}\|$ as in the theorem 6.2.8.

Take $X' = [\rho(A)Y]^-$ and $\pi' = \rho/X'$ i.e., $\pi(a) = P \circ \rho(a) \circ P$ $\forall$ $a \varepsilon A$, where P is the projection as in the proposition 6.2.13.

Now, by proposition 6.2.14, $\rho \sim \pi'$ and therefore, $\pi \sim \pi'$ and π' is non-degenerate.

Let $\xi = P(\tilde{\xi})$ and $\eta = \tilde{\eta}$. Then $\xi \varepsilon X'$; $\eta \varepsilon (X')^*$.
It is clear that $f = (\pi'(a)\xi, \eta)$.
Since $\pi \sim \pi'$, $\|f\|_\pi = \|f\|_{\pi'}$,

$$\|\tilde{\xi}\| \, \|\eta\| = \|\tilde{\xi}\| \, \|\tilde{\eta}\| = \|f\|_\pi = \|f\|_{\pi'}$$

$$\leq \|P(\tilde{\xi})\| \, \|\eta\| \leq \|\tilde{\xi}\| \, \|\eta\|$$

Thus, $\|f\|_\pi = \|\xi\| \, \|\eta\|$.

Hence the corollary.

6.2.16 Corollary:

If A is a Banach algebra lattice, and (π, X) a p-representation then so is (π', X'), where (π', X') is as in the above corollary.

Proof:

It is enough to observe that if Y is a L^p-space and P is a projection on it with $\|P\| = 1$, then the image is also an L^p-space. But this is true. (See Lacey [29, Chapter 7]).

Section 6.3:

Let G be a locally compact group. As usual we make it a convention that for every B-space representation (π,X) of G we have, $\|\pi(t)\| = 1$ for each $t\in G$ (Cf. 1.2.4). Therefore, the corresponding representation of $L^1(G)$ which is again denoted by (π,X), will satisfy $\|\pi(f)\| \leq \|f\|$ $\forall$ $f\in L^1(G)$.

6.3.1a Definition:

We say that a B-space representation (π,X) of G is a lattice representation if,

(i) X is a B-lattice.

(ii) $\forall$ $t\in G$, $\pi(t)$ is a lattice isomorphism.

A lattice representation (π,X) of G is called a p-representation if X is a p-space; $1<p<\infty$.

Remark:

Notice that any lattice representation of G corresponds to a non-degenerate, lattice representation of the Banach algebra lattice $L^1(G)$, and all the non-degenerate lattice representations of $L^1(G)$ arise only this way. And this is the case for p-representations also.

6.3.1b Definition:

If $\mathcal{R}$, is a collection of B-space representations and π is another B-space representation of G then we say that π is weakly contained in $\mathcal{R}$ if π is weakly contained in $\mathcal{R}$ when π and $\mathcal{R}$ are considered as representations of B-algebra $L^1(G)$.

Similarly for quasi-equivalence among the B-space representations of G.

6.3.2 Theorem:

Let (π, X) be a B-space representation of G, with X a reflexive space. Then there exists a representation (π', X') of G such that,

(i) $\pi \sim \pi'$ and

(ii) if $\varphi \in W_\pi$, then there exist $\xi \in X'$; $\eta \in (X')^*$ such that $\varphi(t) = (\pi'(t)\xi, \eta)$ locally almost every where.

Moreover, if

(a) if π is a lattice representation then so is π'.

(b) if π is a p-representation then so is π' ; $1<p<\infty$.

Proof:

Applying corollaries 6.2.15 and 6.2.16, we get the results.

Let $1<p<\infty$ be fixed. Let $\mathcal{P}$ denote the collection of all p-representations of G.

6.3.4 Theorem:

$W_{\mathcal{P}}$ is a commutative Banach algebra of functions over G with unit.

6.3.5 Lemma:

Let (π_1, X_1); (π_2, X_2) be any two p-representations, then $\pi_1 \otimes \pi_2$ is also a p-representation.

(Refer 1.1.3 for the definition of $X_1 \otimes_\ell X_2$).

Proof:

We observe that $X_1 \otimes_{\ell} X_2$ is a p-space with the positive cone given by the $\overline{\{\text{convex cone spanned by } (X_1)_+ \otimes (X_2)_+.\}}$

For every $t \varepsilon G$ define $(\pi_1 \otimes \pi_2)(t)$ by

$$(\pi_1 \otimes \pi_2)(t)(\xi_1 \otimes \xi_2) = \pi_1(t)\,\xi_1 \otimes \pi_2(t)\xi_2$$

for every $\xi_1 \varepsilon X_1$; $\xi_2 \varepsilon X_2$.

Using the fact that $\pi_1(t)$ and $\pi_2(t)$ are lattice isomorphisms and the definition of the positive cone of $X_1 \otimes_{\ell} X_2$ given above, we observe that $\pi_1 \otimes \pi_2(t)$ is also a lattice isomorphism.

Hence the lemma.

Now we prove the theorem 6.3.4.

By the proposition 6.2.7, $W_{\mathcal{P}} = W_{\pi}$ for some B-space representation π. Since $\mathcal{P}$ is a collection of p-representations π is also a p-representation.

Now W_{π} is already seen to be a B-space.

Claim:

W_{π} is an algebra.

Let φ_1, φ_2 be in W_{π}. Then by theorem 6.3.3, there exists another representation $(\tilde{\pi}, \mathfrak{X})$ such that $W_{\pi} = W_{\tilde{\pi}}$ and there exist ξ_1, ξ_2 in $\mathfrak{X}$ and η_1, η_2 in $(\mathfrak{X})^*$ satisfying the following.

(a) $\varphi_1 = \tilde{\pi}_{\xi_1,\eta_1}$ and $\varphi_2 = \tilde{\pi}_{\xi_2,\eta_2}$ locally almost everywhere.

(b) $\|\varphi_1\|_{\mathcal{B}} = \|\varphi_1\|_{\pi} = \|\xi_1\| \, \|\eta_1\|$ and

$\|\varphi_2\|_{\mathcal{B}} = \|\varphi_2\|_{\pi} = \|\xi_2\| \, \|\eta_2\|$.

Now consider the p-representation $(\tilde{\pi} \otimes \tilde{\pi}, \tilde{X} \otimes_{\ell} \tilde{X})$. If we denote this representation by ρ, then we have,

$$\rho_{\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2} = \varphi_1 \cdot \varphi_2 \quad \text{for,}$$

$$\begin{aligned}
\rho_{\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2}(t) &= (\rho(t)\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2) \\
&= (\tilde{\pi}(t)\xi_1 \otimes \tilde{\pi}(t)\xi_2, \eta_1 \otimes \eta_2) \\
&= (\tilde{\pi}(t)\xi_1, \eta_1)(\tilde{\pi}(t)\xi_2, \eta_2) \\
&= \varphi_1(t) \cdot \varphi_2(t), \quad \forall t \in G.
\end{aligned}$$

Thus $\varphi_1 \cdot \varphi_2$ is in $W_{\mathcal{B}}$.

By our constructions we infer that $\pi, \tilde{\pi}, \rho$ are all in .

<u>Claim</u>:

$\|\varphi_1 \cdot \varphi_2\|_{\mathcal{B}} \leq \|\varphi_1\|_{\mathcal{B}} \|\varphi_2\|_{\mathcal{B}}$.

Let $f \in L^1(G)$, then we have,

$$\begin{aligned}
|(f, \varphi_1 \cdot \varphi_2)| &= |(\rho(f)(\xi_1 \otimes \xi_2), (\eta_1 \otimes \eta_2))| \\
&\leq \|(\xi_1 \otimes \xi_2)\| \, \|(\eta_1 \otimes \eta_2)\| \, \|\rho(f)\| \\
&\leq \|\xi_1\| \, \|\xi_2\| \, \|\eta_1\| \, \|\eta_2\| \sup_{\sigma \in \mathcal{B}} \|\sigma(f)\|.
\end{aligned}$$

Therefore,

$$\|\varphi_1\cdot\varphi_2\|_{\mathcal{P}} \le \|\xi_1\|\ \|\eta_1\|\ \|\xi_2\|\ \|\eta_2\|$$

$$= \|\varphi_1\|_\pi\cdot\|\varphi_2\|_\pi$$

$$= \|\varphi_1\|_{\mathcal{P}}\cdot\|\varphi_2\|_{\mathcal{P}}$$

Hence, $W_{\mathcal{P}}$ is a commutative B-algebra.

Let ρ be the trivial representation of G on a p-space X. If $\xi \varepsilon X$ and $\eta \varepsilon X^*$ such that $(\xi,\eta) = 1$ then, we observe that $\rho_{\xi,\eta}$ is the constant function on G with value 1 on G.

This function will serve as a unit for this B-algebra.

Hence the theorem.

Remark:

For every fixed $1<p<\infty$, we denote the above B-algebra by $F_p(G)$.

6.3.6 Remarks:

(i) Let $1<p<\infty$. Then the Figa-Talamanca -Herz algebra $A_p(G)$ is a sub-algebra of $F_p(G)$. (Refer 1.2.3 for the discussion of $A_p(G)$).

Observe that $A_p(G) \subseteq W_{\lambda_p}$ where λ_p is the p-regular representation of G on $L^p(G)$. (Refer 6.1.2).

Moreover $\lambda_p \varepsilon \mathcal{P}$ and therefore, $A_p(G) \subseteq W_{\lambda_p} \subseteq W$.

Since $A_p(G)$ is an algebra with point-wise multiplication, $A_p(G)$ is infact a subalgebra of $F_p(G)$.

(ii) The Banach algebra $W_p(G)$ defined by Cowling [7, Section 4, Theorem 5] is actually W_{λ_p} in our notation. Furthermore, $W_p(G)$ is a subalgebra of $F_p(G)$ for, the multiplication defined in $W_p(G)$ is also pointwise multiplication of functions.

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